# Strategic trading and unobservable information acquisition

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**Abstract**

We allow a strategic trader to choose when to acquire information about an asset's payoff, instead of endowing her with it. When the trader dynamically controls the precision of a flow of information, the optimal precision evolves stochastically and increases with market liquidity. However, because the trader exploits her information gradually, the equilibrium price impact and market uncertainty are unaffected by her rate of acquisition. Instead, if she pays a fixed cost to acquire "lumpy" information at a time of her choosing, the market can break down: we show there exist no equilibria with endogenous information acquisition. Our analysis suggest caution when applying insights from standard strategic trading models to settings with information acquisition.

JEL: D82, D84, G12, G14

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# 1 Introduction

The canonical strategic trading framework, introduced by Kyle (1985), is foundational for understanding how markets incorporate private information. The vast literature that builds on this framework provides many important insights into how informed investors trade strategically on their private information in a variety of market settings and the consequences for asset prices.[1] The framework has also been used to guide a large body of empirical analysis and policy recommendations about liquidity, price informativeness, market design, and disclosure regulation.

However, a key limitation of this setting is that the strategic trader is endowed with private information before trading begins, instead of acquiring it endogenously at a time of her choosing. This assumption is restrictive because the value of acquiring private information can change over time and with economic conditions. In particular, information about an asset's payoff is likely to be more valuable when (i) fundamental uncertainty is higher and (ii) speculative trading opportunities are more attractive (e.g., if uninformed, "noise" trading in the asset increases).[2] As such, it is natural that investors optimize not only how they trade on their private information, but also when they acquire such information.

To study this behavior, we extend the continuous-time Kyle (1985) framework to allow for unobservable, costly information acquisition. Section 2 introduces the model. A single risky asset is traded by a risk-neutral, strategic trader and a mass of noise traders. The asset payoff is publicly revealed at a random time.[3] We allow the volatility of noise trading to evolve stochastically over time (e.g., as in Collin-Dufresne and Fos, 2016). This captures the notion that the profitability of trading opportunities, and consequently, the value of acquiring information for the trader evolves over time. A risk-neutral market maker competitively sets the asset's price, conditional on aggregate order flow and public information.

In contrast to the previous literature, we do not assume the strategic trader is endowed with private information. Instead, she can choose to acquire information in one of two ways. Section 3 considers settings where the trader can acquire information "smoothly" — she dynamically chooses the precision of a flow of signals, subject to a flow cost of precision. This specification captures settings in which traders can gradually scale up or

[1]The literature is truly vast — as of July 2019, the paper has over 10,400 citations on Google Scholar.

[2]The latter channel is consistent with the observation that institutional investors often concentrate their trading in high volume securities, and sometimes even pay exchanges to trade against retail order flow.

[3]The assumption of a random horizon is largely for tractability and is not qualitatively important for our primary results. What is key is that a random horizon induces the trader to discount future profits. We expect our results to carry over to settings with fixed horizon that feature discounting for other reasons (e.g., if the trader has a subjective discount factor or the risk-free rate is nonzero). Section 4.2.5 discusses how our results are affected when there is no discounting.

down the attention or scrutiny they pay to their sources of information e.g., by focusing on a specific stock or industry in response to sudden increases in turnover or specific news events. In contrast, Section 4 considers the case of "lumpy" acquisition costs — the trader can acquire discrete, payoff relevant signals at a fixed cost (á la Grossman and Stiglitz (1980)), but optimally chooses the timing of this acquisition. This captures situations in which information acquisition involves fixed costs e.g., hiring new analysts, investing in new technology, or conducting research about a new market or asset class. Crucially, in either case, we assume the market maker cannot observe or immediately detect the trader's information acquisition choices — instead, this must be inferred from the observed order flow.[4]

Our results show that the type of information acquisition technology affects not only the nature of equilibrium, but its very existence. When information costs are smooth, we show that the trader's optimal choice of precision evolves stochastically with trading opportunities, and is higher when uninformed trading volatility and market liquidity are higher. However, because the trader optimally exploits her informational advantage smoothly, the equilibrium price impact and market uncertainty are invariant to her the rate at which she acquires information. In contrast, when information costs are lumpy, dynamic information acquisition generally leads to equilibrium breakdown. We show that there cannot exist any Markovian equilibria with endogenous information acquisition and strategic trading, including those in which the trader can employ mixed acquisition strategies.[5]

The analysis in Section 3 implies that allowing for endogenous information acquisition has novel implications for the dynamics of market liquidity and how well prices reflect information about fundamentals. As is common in the literature, we capture market liquidity using price impact (i.e., Kyle's $\lambda$). Moreover, following Weller (2017), we study the behavior of two related but distinct notions of how well prices reflect information about fundamentals: *price informativeness*, which is an absolute measure of the *total* information content of prices, and *informational efficiency*, which is a relative measure of the *fraction* of the strategic trader's

[4]As we shall see, the unobservability of acquisition plays an important role. In a companion paper (Banerjee and Breon-Drish, 2018), we consider a setting where entry (and implicitly the associated information acquisition) is publicly observable, and explore the implications of allowing the strategic trader to choose when to enter a new trading opportunity.

[5]Our non-existence result is more general, and as we show, applies in any setting in which so-called "trade-timing indifference" holds. We focus attention on Markovian settings for analytical tractability in establishing trade-timing indifference and for consistency with prior work. Moreover, we allow the equilibrium strategies to depend on an arbitrarily large (though finite) number of state variables. As such, the focus on Markov equilibria it is less restrictive than it may initially appear. The class of mixed strategies we rule out include equilibria that involve "discrete" mixing in which the trader acquires at a countable collection of times and/or "continuous" mixing in which the trader acquires information with a given intensity over some interval of time.

private information revealed by prices.[6] When the trader is endowed with information, as in standard strategic trading models, more precise private information is associated with higher price impact and, consequently, lower market liquidity. Moreover, both price informativeness and informational efficiency increase over time as the strategic trader gradually loses her informational advantage over the market maker.

These predictions do not hold when information acquisition is endogenous. First, we show that the trader optimally increases the precision of her signal when noise trading volatility is high and price impact is low, because this is when trading on private information is more valuable. However, this implies that adverse selection and liquidity can be *positively* related in our model: shocks to uninformed trading volatility simultaneously lead to higher liquidity (lower $\lambda$) and higher information acquisition. These predictions are broadly consistent with the evidence documented by Ben-Rephael, Da, and Israelsen (2017), who show that their measure of abnormal institutional investor attention is higher on days with higher abnormal trading volume, and Drake, Roulstone, and Thornock (2015) who show that EDGAR search activity is higher following days with high turnover.[7]

Second, we show that the equilibrium price impact and market uncertainty are unaffected by the rate at which the trader acquires information. Because she can strategically respond to changes in price impact by trading more or less aggressively, she trades gradually to maximize the value of her private information. As a result, even though the trader's rate of information acquisition depends on trading opportunities (e.g., noise trading volatility), the rate at which the market maker learns about fundamentals does not. An immediate consequence is that informational efficiency and price informativeness can move in opposite directions when information acquisition is endogenous. While price informativeness always increases over time, informational efficiency can decrease when the trader acquires private information more quickly than the market maker learns from order flow. Moreover the disconnect between price efficiency and informativeness is more likely when uninformed trading volatility is high and the trader faces more uncertainty about the asset payoff. Importantly, this set of predictions flows naturally from the strategic behavior of the trader and distinguishes our model both from ones with exogenous information endowment, and from models of dynamic acquisition

[6] More precisely, we define price informativeness as the percentage reduction in the market maker's posterior variance by date $t$, and informational efficiency as the ratio of the market maker's posterior precision about payoffs to the strategic trader's posterior precision.

[7] While information available on Bloomberg and EDGAR is nominally public, since at least Kim and Verrecchia (1994), it has been recognized that a natural source of "private information" for a trader is a superior ability to process public information. Hence, in the recent literature on search activity it is common to interpret searches as a proxy for private information acquisition. There is evidence that identifying private information with superior processing of public information is reasonable. For instance, in the context of short-selling, Engelberg, Reed, and Ringgenberg (2012) empirically document that short-sellers' information advantage is, in large part, due to their skill in utilizing publicly-available information.

that feature competitive trading (e.g., Han (2018)).

The non-existence results of Section 4 further emphasize the importance of carefully considering endogenous information acquisition, and specifically, how one models it. Our analysis uncovers two key economic forces that apply to settings in which information costs are lumpy and which generally lead to equilibrium breakdown. First, if a trader can acquire information earlier than the market anticipates without being detected, she can exploit her informational advantage by trading against a pricing rule that is insufficiently responsive. We refer to this as a *pre-emption deviation* and show that it rules out any equilibria in which she acquires information after the initial date. While particularly transparent in our setting, this force is likely to rule out pure-strategy acquisition equilibria with delay in more general settings (e.g., with multiple strategic or non-strategic traders).

Second, because she optimally smoothes her trades over time, a strategic trader's gains from being informed over any *short* period are small when trading opportunities are frequent. In continuous time this leads to "trade-timing indifference" — over any finite interval of time, an informed strategic trader is indifferent between trading along her equilibrium strategy or refraining from trade over that interval and then trading optimally going forward.[8] This implies that instead of acquiring information at the time prescribed in (any conjectured) equilibrium, the trader can instead wait over an interval, and then acquire information. This *delay deviation* is strictly profitable since she does not incur a loss in expected trading gains, but benefits (in present value) by delaying the cost of information acquisition.

The analysis in Appendix B highlights that our non-existence result rule out a large class of equilibria. Moreover, as we discuss in Section 4.2, trade timing indifference and the delay deviation is pervasive: it arises naturally under more general assumptions about the market maker's preferences, the trader's risk-aversion and information endowment, the frequency of trading (i.e., when trading is frequent, but not continuous), and in the absence of discounting.

Section 5 concludes by discussing some implications for future work. Our analysis highlights the importance of explicitly accounting for endogenous information acquisition in models of strategic trading. The trading equilibrium with endogenous acquisition, when it exists, is qualitatively different from one in which investors are endowed with private information. This is particularly important for empirical and policy analysis: one cannot simply interpret existing models as reduced form variants of models with endogenous information

[8]This property of the standard Kyle (1985) framework was first noted by Back (1992). However, it also applies to settings in which the strategic trader is risk-averse, or the market maker is risk averse and/or not perfectly competitive. Economically, timing indifference arises because, in equilibrium, there cannot be any predictability in the level or slope of the price function if the trader refrains from trading. If there were, then she could deviate from any proposed trading strategy by waiting and then exploit such predictability.

acquisition. Furthermore, the sharp contrast in conclusions across the smooth and lumpy cost cases emphasizes that the choice of the information acquisition technology is not simply a matter of modeling convenience or tractability, but has important consequences for equilibrium outcomes.

## 1.1 Related Literature

Our paper is at the intersection of two closely related literatures. The first follows Kyle (1985) and focuses on strategic trading by investors who are exogenously endowed with information about the asset payoff (e.g., see Back (1992), Back and Baruch (2004), Caldentey and Stacchetti (2010), and Collin-Dufresne and Fos (2016)). The second follows Grossman and Stiglitz (1980) and studies endogenous information acquisition in financial markets. However, in contrast to our setting, these papers restrict investors to acquire information (or commit to a sequence of signals) before the start of trade.[9]

Two notable exceptions are Banerjee and Breon-Drish (2018) and Han (2018). A companion to the current paper, Banerjee and Breon-Drish (2018) also consider a strategic trading environment but study dynamic information acquisition and entry that is detectable by the market maker. Importantly, in settings when information costs are lumpy but entry is observable, they show that equilibria can be sustained — moreover, the model features delayed entry and information acquisition by strategic traders.

Han (2018) considers a dynamic model in which heterogeneously informed, competitive investors (á la Hellwig (1980), as generalized by Breon-Drish (2015)) dynamically choose to allocate attention in response to changes in aggregate uncertainty. He shows that investors acquire more precise information when uncertainty is high (as in our smooth acquisition case), but this feeds back through prices to lower uncertainty in future periods. This contrasts with our setting in which the rate at which the trader learns is higher when her uncertainty is high, but future public (i.e., market maker's) uncertainty is unaffected by the rate of information acquisition. This is due to the fact that, in our setting, the strategic trader optimally smoothes her use of information over time. This illustrates an important difference between dynamic information acquisition in models of strategic versus competitive trading.

Finally, since the market maker does not know how informed the strategic trader is, our paper is also related to a recent literature that studies markets in which some participants face uncertainty about the existence or informedness of others (e.g., Chakraborty and Yılmaz

[9]For instance, Back and Pedersen (1998) and Holden and Subrahmanyam (2002) allow investors to precommit to receiving signals at particular dates, while Kendall (2018), Dugast and Foucault (2018), and Huang and Yueshen (2018) incorporate a time-cost of information. Veldkamp (2006) considers a sequence of one-period information acquisition decisions. In all these cases, the information choices occur before trading and so acquisition is effectively a static decision.

(2004), Alti, Kaniel, and Yoeli (2012), Li (2013), Banerjee and Green (2015), Back, Crotty, and Li (2017), Dai, Wang, and Yang (2019)). Often, uncertainty about whether others are informed can be nested in a more general model where investors learn not only about fundamentals, but also about the information that others have. However, this need not always be the case — as the analysis in Section 4 highlights, this higher order uncertainty can lead to equilibrium non-existence market breakdown in some settings.[10]

# 2 Model Setup

Our framework is based on the continuous-time Kyle (1985) model with a random horizon (as in Back and Baruch (2004) and Caldentey and Stacchetti (2010)), generalized to allow for stochastic volatility in noise trading as in Collin-Dufresne and Fos (2016).[11] There are two assets: a risky asset and a risk-free asset with interest rate normalized to zero. The risky asset pays off a terminal value $V \sim N(0, \Sigma_0)$ at random time $T$, where $T$ is exponentially distributed with rate $r$ and independent of $V$.[12] We assume that all market participants have common priors over the distribution of payoffs and signals.

There is a single, risk-neutral strategic trader. Let $X_t$ denote the cumulative holdings of the trader, and suppose the initial position $X_0 = 0$. In addition to the strategic trader, there are noise traders who hold $Z_t$ shares of the asset at time $t$, where

$$dZ_t = \nu_t \, dW_{Zt}, \tag{1}$$

where noise trading volatility $\nu_t$ can follow a general, positive stochastic process. Specifically, we assume

$$\frac{d\nu_t}{\nu_t} = \mu_\nu(t, \nu_t) \, dt + \sigma_\nu(t, \nu_t) \, dW_{\nu t}, \tag{2}$$

where $W_{Zt}$ and $W_{\nu t}$ are independent Brownian motions, and the coefficients are such that there exists a unique, strong solution to the stochastic differential equation.[13] Moreover, we

[10] Dai et al. (2019) show an analogous result when a risk-neutral, competitive market maker faces uncertainty about the existence of an (exogenously) informed, strategic trader. They show that equilibrium existence can be restored when the market maker is a monopolist. In our setting, because the strategic trader chooses when to acquire information, non-existence can obtain even when the market maker is a monopolist.

[11] We introduce stochastic volatility so that the value of information varies over time in a non-trivial, but tractable and economically reasonable, way. In a previous version of this paper, we showed that similar results hold when there is an ongoing flow of public news about the asset value itself.

[12] While we consider the case of a fixed asset value $V$ for comparability to previous work, there is no difficulty in accommodating an asset value that evolves over time as $dV_t = (a(t) - b(t)V_t)dt + \sigma_V(t)dW_{Vt}$ for some general, deterministic functions $a, b$, and $\sigma_V$ and independent Brownian motion $W_{Vt}$. Furthermore, none of our results differ qualitatively in such a setting. It is also straightforward to extend our results to allow for a general, continuous distribution for $T$.

[13] For concreteness, we specify that the shocks to $\nu$ are Brownian and that the $\nu$ process is Markov, but

assume that $\nu_t$ is publicly observable to all market participants. This is without loss since $\nu_t$ enters all relevant equilibrium expressions only through $\nu_t^2$, which is the equilibrium order flow volatility and can be inferred perfectly from the realized quadratic variation of order flow.

The key difference from the existing literature is that the trader is not endowed with information about $V$. Instead, we assume that she can acquire costly information about $V$ at a time of her choosing. Specifically, we will separately consider two types of information technology available to the trader. In Section 3, the cost of information acquisition is "smooth": the trader observes a flow of signals $\{S_t\}$, and chooses the (instantaneous) precision of these signals, $\eta_t$, optimally, subject to a flow cost incurred at a rate $c(\eta)\,dt$. This captures settings in which traders can dynamically adjust the level of attention or scrutiny they pay to information available to them. In Section 4, we assume the cost of acquisition is "lumpy": the trader has access to a (discrete) signal $S_0$ at fixed cost $c > 0$, but the trader optimally chooses the time $\tau$ at which to acquire it. This approximates settings in which information acquisition involves a fixed cost e.g., to conduct new research, hire additional analysts or invest in new technology. We formally describe the assumptions about acquisition costs in Sections 3 and 4, respectively.

A competitive, risk neutral market maker sets the price of the risky asset equal to the conditional expected payoff given the public information set. Let $\mathcal{F}_t^P$ denote the public information filtration, which is that induced from observing the aggregate order flow process $Y_t = X_t + Z_t$ and stochastic noise trading volatility $\nu_t$, i.e., $\mathcal{F}_t^P$ is the augmentation of the filtration $\sigma(\{\nu_t, Y_t\})$. Crucially, whether or not the trader has acquired information (or how much information has been acquired) is not directly observable. Rather, as part of updating his beliefs about the asset value, the market maker must also use the public information to infer how informed the trader is. The price at time $t < T$ is given by

$$P_t = \mathbb{E}\left[V \middle| \mathcal{F}_t^P\right]. \tag{3}$$

Let $\mathcal{F}_t^I$ denote the augmentation of the filtration $\sigma(\mathcal{F}_t^P \cup \sigma(\{S_t\}))$. Thus, $\mathcal{F}_t^I$ represents an informed trader's information set. Following Back (1992), we require an *admissible* trading strategy to be a semi-martingale adapted to the trader's filtration, which is a minimal condition for stochastic integration with respect to $X$ to be well-defined. That is, (i) in the case of "smooth" information acquisition, her strategy must be adapted to $\mathcal{F}_t^I$, and (ii) in the case of "lumpy" information acquisition her pre-acquisition trading strategy must be adapted to $\mathcal{F}_t^P$ and her post-acquisition strategy adapted to $\mathcal{F}_t^I$. Our definition of equilibrium

there is no difficulty in extending the shocks to be general martingales and allowing for history-dependence as in Collin-Dufresne and Fos (2016).

is standard, but modified to account for endogenous information acquisition.

**Definition 1.** *An equilibrium is (i) an information acquisition strategy and an admissible trading strategy $X_t$ for the trader and (ii) a price process $P_t$, such that, given the trader's strategy the price process satisfies* (3) *and, given the price process, the trading strategy and acquisition strategy maximize the expected profit*

$$\mathbb{E}\left[(V - P_T)\, X_T + \int_{[0,T]} X_{u^-} dP_u\right]. \tag{4}$$

Note that in the case of "smooth" costs, the information acquisition strategy is a $\mathcal{F}_t^I$-measurable process $\eta_t \geq 0$, while in the case of "lumpy" costs, the information acquisition strategy is a probability distribution over the set of $\mathcal{F}_t^P$-measurable stopping times.[14] We will place additional structure on the class of admissible trading strategies and information acquisition strategies in the following Sections.

Finally, we let $J(t, \cdot)$ denote the value function for a trader of type $s$ and note that it can be expressed as

$$\begin{aligned} J(t, \cdot) &= \mathbb{E}\left[(V - P_T)\, X_T + \int_{[t,T]} X_{u^-} dP_u\right] \\ &= \mathbb{E}\left[\int_{[t,T]} (V - P_{u^-})\, dX_u + [X, V - P]_{[t,T]}\right] \\ &= E\left[\int_{[t,\infty)} e^{-r(u-t)} (V - P_{u^-})\, dX_u + \int_{[t,\infty)} e^{-r(u-t)} d[X, V - P]_u\right] \end{aligned}$$

where the first equality follows from the integration by parts formula for semi-martingales and the second from the fact that $T$ is independently exponentially distributed.

## 2.1 Discussion of Assumptions

Our model assumptions make the analysis in Section 3 tractable and allow us to compare our results to the existing literature in a unified setting. However, our non-existence results in Section 4 apply to more general settings in which we relax most of the above assumptions on the payoff structure and the process for noise trade. For instance, here we focus on stochastic volatility as the key determinant of the trader's information choice, because it is a natural, empirically relevant channel through which the value of acquiring information changes over time. Similarly, the assumption that the asset payoff $V$ is normally distributed

[14] As we discuss further in Section 4, this notion of lumpy information acquisition allows the trader to follow mixed acquisition strategies by randomizing over stopping times in $\mathcal{T}$.

is standard in the literature, and provides a natural benchmark. As we describe in Appendix B, however, the non-existence results of Section 4 obtain even when we allow the risky payoff to evolve over time and be a general (sufficiently smooth) function of the private information of the trader and an arbitrary number of publicly observable signals (news). Moreover, as we discuss in Section 4.2, similar non-existence results also obtain under alternative assumptions about the market maker's objective function, the trader's preferences, discreteness in trading opportunities, and (non)discounting of future payoffs.

To isolate and emphasize the implications of *endogenous* information acquisition, we follow the literature by assuming that the market participants begin with common priors, and the trader is not endowed with any private information. In practice, traders may be endowed with some private information in addition to acquiring costly information. However, we expect that our results would be qualitatively similar in settings where the trader is initially endowed with certain types of payoff-relevant, private information. For instance, the analysis in Section 3 can allow for an initial lump of private information, with either exogenous or endogenous precision, in the form of a conditionally normal signal about $V$. Our key results on the dynamics of liquidity and market uncertainty would be qualitatively identical in such a setting. Similarly, as we discuss further in Section 4.2.3, we expect the key economic forces that lead to non-existence in the case of lumpy costs also apply in settings where the strategic trader is initially endowed with some private information about the asset payoff. On the other hand, allowing for private information that is not directly relevant, but may affect the trader's acquisition strategy (e.g., about the acquisition cost or the trader's preferences) is substantially more challenging and beyond the scope of the current paper.

# 3 Smooth Acquisition

In this section, we consider a setting in which the information acquisition cost is smooth.[15] Specifically, we assume the trader can observe a flow of signals of the form

$$dS_t = V dt + \sqrt{\frac{1}{\eta_t}} dW_{st}, \tag{5}$$

where $W_{st}$ is a standard Brownian motion, independent of $W_{\nu t}$ and $W_{Zt}$ and $\eta_t$ is the instantaneous precision of the signal process. Given her information set $\mathcal{F}_t^I = \sigma(\{S_s, Y_s\}_{0 \leq s \leq t})$, the trader dynamically chooses the precision process $\eta_t$, subject to a flow cost incurred at a rate $c(\eta)\, dt$. We assume that the cost function $c : [0, \infty) \to [0, \infty)$ is twice continuously differentiable with $c \geq 0$, $c' \geq 0$, $c'' > 0$. Moreover, we assume that $c'(0) = 0$ and

[15] We thank Dmitry Orlov for suggesting that we explore this approach.

$\lim_{\eta\to\infty} c'(\eta) = \infty$, which ensures that $c'(\eta)$ has a well-defined, continuously differentiable inverse function $f(\cdot) \equiv [c']^{-1}(\cdot)$ defined on all of $[0, \infty)$.

## 3.1 Characterizing the equilibrium

Our analysis of the equilibrium largely follows that in Collin-Dufresne and Fos (2016), generalized where necessary for endogenous precision choice. Denote the trader's conditional beliefs about the payoff $V$ by $V_t \equiv \mathbb{E}\left[V \middle| \mathcal{F}_t^I\right]$ and $\Omega_t \equiv \text{var}\left[V \middle| \mathcal{F}_t^I\right]$. Denote the market maker's conditional beliefs *about the trader's value estimate* by $\bar{V}_t \equiv \mathbb{E}\left[V_t \middle| \mathcal{F}_t^P\right]$ and $\Psi_t \equiv \text{var}\left[V_t \middle| \mathcal{F}_t^P\right]$ and denote his conditional variance of the asset value itself as $\Sigma_t = \text{var}\left[V \middle| \mathcal{F}_t^P\right]$. Note that by the law of iterated expectations, $\overline{V}_t$ is also the market maker's conditional expectation of the asset value itself and that by the law of total variance $\Sigma_t = \Omega_t + \Psi_t$.

Following the literature, we will search for an equilibrium in which (i) the optimal trading strategy is of the form:

$$dX_t = \theta_t dt, \text{ where } \theta_t = \beta_t (V_t - P_t), \tag{6}$$

where $\beta_t$ is a $\mathcal{F}_t^P$-adapted process, (ii) the optimal precision choice $\eta_t$ is a $\mathcal{F}_t^P$-adapted process,[16] (iii) the pricing rule is

$$dP_t = \lambda_t dY_t, \tag{7}$$

where $\lambda_t$ is a $\mathcal{F}_t^P$-adapted process, and (iv) both the trader and market maker learn the asset value if the economy continues indefinitely (i.e., $\Omega_t \to 0, \Sigma_t \to 0$).[17] We require that the trading strategy satisfies a standard admissibility condition to rule out doubling-type strategies that accumulate unbounded losses followed by unbounded gains:

$$E\left[\int_0^\infty e^{-rs}\theta_s^2 ds\right] < \infty. \tag{8}$$

We need to show that (i) given the conjectured trading and information acquisition strategy, the optimal pricing rule takes the conjectured form, and (ii) given the conjectured pricing rule, the optimal trading and information acquisition strategy take the conjectured form. We proceed by first considering the market maker's filtering problem, then the trader's

[16]Note that we are not restricting the trader to "best responses with $\mathcal{F}_t^P$-adapted coefficients". Rather, given the conjectured pricing rule, we establish that the trader's best response is of the given linear form, with both $\beta_t$ and $\eta_t$ being $\mathcal{F}_t^P$-adapted. Hence, even though information acquisition is not directly observable, *in equilibrium* the market maker effectively knows the trader's optimal precision choice at each instant.

[17]Given the class of precision cost functions that we consider, conjecture (iv) can in fact be derived as a result in equilibria that result under conjectures (i)-(iii). However, for expositional clarity and to eliminate notational clutter, we formulate it as a conjecture to be verified.

optimization problem, and then jointly enforcing the optimality conditions derived for each agent.

### 3.1.1 The market maker's problem

The market maker must filter the asset value from the order flow. Because the trader is better informed than the market maker, the law of iterated expectations implies that it is sufficient that he filter the trader's value estimate from order flow. More specifically, the market maker must compute

$$\mathbb{E}\left[V \middle| \mathcal{F}_t^P\right] = \mathbb{E}\left[\mathbb{E}\left[V \middle| \mathcal{F}_t^I\right] \middle| \mathcal{F}_t^P\right] = \mathbb{E}\left[V_t \middle| \mathcal{F}_t^P\right] \tag{9}$$

from observing

$$dY = dX + dZ = \beta_t(V_t - P_t)dt + \nu_t dW_{Zt}.$$

Using standard filtering theory (e.g., see Liptser and Shiryaev (2001), Ch. 12), under the conjectured acquisition strategy, the trader's beliefs evolve according to

$$dV_t = \sqrt{\eta_t}\Omega_t d\hat{W}_{Vt}, \quad \text{and} \quad d\Omega_t = -\eta_t \Omega_t^2 dt, \tag{10}$$

where $d\hat{W}_{Vt}$ is a Brownian motion under the trader's filtration. Under our maintained conjecture (to be verified) that $\eta_t$ is $\mathcal{F}_t^P$-adapted, we can further apply standard filtering theory to conclude that the market's maker's conditional mean and variance, $(\overline{V}_t, \Psi_t)$ of $V_t$, therefore characterized by

$$d\bar{V}_t = \frac{\beta_t \Psi_t}{\nu_t^2}\left(dY_t - \beta_t\left(\bar{V}_t - P_t\right)dt\right), \quad \text{and} \quad d\Psi_t = \left(\eta_t \Omega_t^2 - \frac{\beta_t \Psi_t^2}{\nu_t^2}\right)dt. \tag{11}$$

Enforcing efficient pricing, i.e., $P_t = \overline{V}_t$, and setting coefficients equal to the conjectured pricing rule yields

$$\lambda_t = \frac{\beta_t \Psi_t}{\nu_t^2} \tag{12}$$

with

$$d\Psi_t = \left(\eta_t \Omega_t^2 - \lambda_t^2 \nu_t^2\right)dt. \tag{13}$$

### 3.1.2 The trader's problem

Let $M_t = V_t - P_t$. Given the conjectured pricing rule and the dynamics of the trader's beliefs $(V_t, \Omega_t)$ in equation (10), note that

$$dM_t = dV_t - dP_t = \sqrt{\eta_t}\Omega_t d\hat{W}_{Vt} - \lambda_t\left(\theta_t dt + \nu_t dW_{Zt}\right), \tag{14}$$

where $\hat{W}_{Vt}$ and $W_{Zt}$ are independent Brownian motions under the trader's filtration. Conjecture that the trader's value function is of the form

$$J = \frac{M_t^2 + \kappa_{1t}}{2\lambda_t} + \kappa_{2t} \tag{15}$$

for some locally-deterministic processes $\kappa_{jt}$ with zero instantaneous volatility i.e., $d\kappa_{jt} = k_{jt}dt$, to be determined. The Hamilton-Jacobi-Bellman (HJB) equation implies:

$$0 = \sup_{\theta\in\mathbb{R},\eta>0} \mathbb{E}_t\left[\begin{array}{c} -rJdt + J_{\kappa_1}d\kappa_{1t} + J_{\kappa_2}d\kappa_{2t} + J_M dM_t + J_{(1/\lambda)}d\left(1/\lambda_t\right) \\ +\frac{1}{2}J_{MM}\left(dM_t\right)^2 + J_{M(1/\lambda)}dM_t d\left(1/\lambda\right)_t + \theta_t M_t dt \\ -c\left(\eta\right)dt \end{array}\right] \tag{16}$$

$$= \sup_{\theta\in\mathbb{R},\eta>0} \mathbb{E}_t\left[\begin{array}{c} \left(-r\frac{M_t^2+\kappa_{1t}}{2\lambda_t}dt + \frac{M_t^2+\kappa_{1t}}{2}d\left(1/\lambda_t\right)\right) + \left(\frac{M_t}{\lambda_t}dM_t + \theta_t M_t dt\right) \\ + \left(\frac{1}{2\lambda_t}d\kappa_{1t} + \frac{1}{2}\frac{1}{\lambda_t}\left(dM_t\right)^2\right) + \left(d\kappa_{2t} - \left(r\kappa_{2t} + c\left(\eta\right)\right)dt\right) + M_t dM_t d\left(1/\lambda\right)_t \end{array}\right] \tag{17}$$

where the second equality substitutes for the derivatives of $J$ and the dynamics of the processes, and groups terms. The first order condition with respect to $\theta_t$ is

$$-J_M\lambda_t + M_t = 0, \tag{18}$$

which holds trivially since $J_M = M_t/\lambda_t$. The first order condition with respect to $\eta$ is

$$\tfrac{1}{2}J_{MM}\Omega^2 - c'\left(\eta_t\right) = 0 \tag{19}$$

where $J_{MM} = 1/\lambda_t$. Since $c''\left(\eta\right) > 0$, the second order condition that the Hessian with respect to $(\theta, \nu)$ is negative semi-definite is always satisfied at any interior optimum. Furthermore, since $c'(0) = 0$, the optimal precision is always interior (i.e., the trader never finds it optimal to choose $\eta_t = 0$). So, given the conjectured value function, the optimal choice of precision is given by

$$\eta_t^* = f\left(\frac{\Omega_t^2}{2\lambda_t}\right). \tag{20}$$

Since $\lambda_t$ is $\mathcal{F}_t^P$-adapted, it follows that $\Omega_t$ and the process $\eta_t$ so-defined are as well.

Finally, we will have $\mathbb{E}_t\left[dJ_t + \theta_t M_t dt\right] = 0$ as required in eq. (17) if (i) we can find a process for $\left(\frac{1}{\lambda_t}, \Psi_t\right)$ that satisfies the forward-backward stochastic differential equation:

$$\mathbb{E}_t\left[d\frac{1}{\lambda_t}\right] = \frac{r}{\lambda_t}dt \tag{21}$$
$$d\Psi_t = \left(\eta_t\Omega_t^2 - \lambda_t^2\nu_t^2\right)dt$$

and (ii) processes $\kappa_{jt}$ that satisfy:

$$d\kappa_{1t} = -\left(\eta_t\Omega_t^2 + \lambda_t^2\nu_t^2\right)dt \qquad d\kappa_{2t} = \left(r\kappa_{2t} + c(\eta_t)\right)dt \tag{22}$$

with $\kappa_{jt} \to 0$.

### 3.1.3 Equilibrium conditions

Note that in our setting, $\Sigma_t$ can be expressed as:

$$\Sigma_t = \Psi_t + \Omega_t - \Omega_\infty = \Psi_t + \int_t^\infty \eta_s\Omega_s^2 ds, \tag{23}$$

where the first equality uses the law of total variance $\Sigma = \Psi + \Omega$ and $\Omega_t \to 0$, and the second equality uses the dynamics of $\Omega_t$ to substitute for the integral. Intuitively, this implies that the market maker's conditional variance of $V$ (i.e., $\Sigma_t$) is equal to the trader's forward looking informational advantage, which consists of her current advantage $\Psi_t$ plus the advantage that will accrue to her from future learning $\int_t^\infty \eta_s\Omega_s^2 ds$.

If we define $\lambda_t = e^{-rt}\sqrt{\frac{\Sigma_t}{G_t}}$ for some process $G_t$ to be determined, then the equations for $G_t$ and $\Psi_t$ separate:

$$\sqrt{G_t} = \mathbb{E}_t\left[\int_t^\infty \frac{1}{2}e^{-2rs}\frac{\nu_s^2}{\sqrt{G_s}}ds\right] \tag{24}$$
$$\frac{d\Sigma_t}{\Sigma_t} = -e^{-2rt}\frac{\nu_t^2}{G_t}dt \tag{25}$$

with boundary conditions $\lim_{t\to\infty} G_t = 0$ and $\Sigma_0 = \text{var}(V)$. Intuitively, as in Collin-Dufresne and Fos (2016), $G_t$ is a measure of the amount of future noise trading volatility that is relevant for the trader when formulating her optimal trading strategy. . If one can find such processes then the market depth process $(1/\lambda_t)$ so-defined satisfies (21). This implies the following result.

**Theorem 1.** *Suppose there exists a solution $G_t$ to (24), and suppose $\nu_t$ is uniformly bounded between $\overline{\nu} > \underline{\nu} > 0$, then there exists an equilibrium in which:*

*(i) the equilibrium pricing rule is given by (7), where*

$$\lambda_t = e^{-rt}\sqrt{\frac{\Sigma_t}{G_t}}, \quad \Sigma_t = e^{-\int_0^t e^{-2rs}\frac{\nu_s^2}{G_s}ds}\Sigma_0, \tag{26}$$

*(ii) the optimal precision process is given by $\eta_t = f\left(\frac{\Omega_t^2}{2\lambda_t}\right)$, where $f(\cdot) \equiv [c']^{-1}(\cdot)$, $\Omega_0 = \Sigma_0$, $\Omega_t$ evolves according to (10), and $\Omega_t \to 0$ almost surely,*

*(iii) the optimal trading rule is given by (6), where $\beta_t = \frac{\lambda_t \nu_t^2}{\Psi_t}$, $\Psi_0 = \Sigma_0$, $\Psi_t$ evolves according to (13), and $\Psi_t \to 0$ almost surely,*

*(iv) the informed trader's value function is given by (15), where the $\kappa_{jt}$ are given by:*

$$\kappa_{1t} = \Omega_t + \Sigma_t \tag{27}$$

$$\kappa_{2t} = -E_t\left[\int_t^\infty e^{-r(s-t)}c(\eta_s)ds\right] \tag{28}$$

This theorem characterizes equilibrium assuming existence of a solution $G_t$ to (24). The following Lemma establishes existence under some additional conditions.

**Lemma 1.** *If $\nu_t$ is uniformly bounded between $\underline{\nu} < \overline{\nu}$, then there exists a solution $G_t$ that satisfies $e^{-2rt}\frac{\underline{\nu}}{2r} \leq G_t \leq e^{-2rt}\frac{\overline{\nu}}{2r}$. Furthermore, when there exists a solution $G_t$ (with $\nu$ bounded or not), then we have $G_t = \gamma^2(t, \nu_t)$ for a function $\gamma(\cdot)$ that solves the partial differential equation*

$$\gamma_t + \nu\mu_\nu(t,\nu)\gamma_\nu + \frac{1}{2}\nu^2\sigma_\nu^2(t,\nu)\gamma_{\nu\nu} + \frac{1}{2}\frac{e^{-2rt}\nu^2}{\gamma} = 0, \qquad \lim_{t\to\infty}\gamma(t,\nu) = 0.$$

*If $\nu_t$ has a deterministic drift coefficient, i.e.,*

$$\frac{d\nu_t}{\nu_t} = \mu_\nu(t)dt + \sigma_\nu(t,\nu_t)dW_{\nu t},$$

*which satisfies $\int_t^\infty e^{\int_t^s 2(\mu_\nu(u)-r)du}ds < \infty$ for all $t \geq 0$, then the process $G_t$ is available in closed form, i.e., $G_t = B(t)\nu_t^2$, where*

$$B(t) = e^{-2rt}\int_t^\infty e^{\int_t^s 2(\mu_\nu(u)-r)du}ds.$$

A few comments about the Lemma are in order. First, the uniform upper and lower bounds on $\nu_t$ are sufficient to guarantee existence of a solution, but are not necessary. In particular, the case with deterministic drift does not necessarily satisfy either of them, but we are able to construct a solution in closed-form as in the Lemma. Second, assuming that a solution $G_t$ exists, a uniform lower (upper) bound on $\nu_t$ ensures the lower (upper) bound on $G_t$. That is, we only need to bound $\nu_t$ from one side in order to bound $G_t$ from the same side. Third, the PDE representation for $G_t$ holds when there is a solution, bounded or not, and relies on the fact that the $\nu$ process is Markov. Finally, while the deterministic-drift case may seem a bit artificial at first glance, it nests natural benchmark cases of (i) geometric Brownian motion (i.e., $\mu_\nu(t) \equiv \mu_\nu$, $\sigma_\nu(t, \nu_t) \equiv \sigma_\nu$) and (ii) general martingale dynamics (i.e., $\mu_\nu(t) \equiv 0$).

## 3.2 Implications of endogenous information acquisition

We now show that allowing for endogenous information acquisition has important implications for price impact, informational efficiency and the total informativeness of prices. Moreover, our analysis suggests that one must be careful in interpreting patterns in observable return-volume characteristics as evidence for unobservable information acquisition.

We begin with a characterization of price impact, or Kyle's $\lambda$, in our setting.

**Corollary 1.** *Suppose there exists a solution $G_t$ to (24), and let $\gamma(t, \nu_t) = \sqrt{G_t}$. The evolution of market depth $\frac{1}{\lambda_t}$ is*

$$\frac{d(\frac{1}{\lambda_t})}{\frac{1}{\lambda_t}} = r dt + \nu_t \sigma_\nu(t, \nu_t) \frac{\gamma_\nu}{\gamma} dW_{\nu t}.$$

*Therefore, the evolution of price impact $\lambda_t$ is*

$$\frac{d\lambda_t}{\lambda_t} = \left( \nu^2 \sigma_\nu^2(t, \nu_t) \frac{\gamma_\nu^2}{\gamma^2} - r \right) dt - \nu_t \sigma_\nu(t, \nu_t) \frac{\gamma_\nu}{\gamma} dW_{\nu t}. \tag{29}$$

*Moreover, if $\nu_t$ has a deterministic drift coefficient, i.e., $\mu_\nu(t, \nu) = \mu_\nu(t)$, then the evolution of $\lambda$ simplifies to*

$$\frac{d\lambda_t}{\lambda_t} = \left( \sigma_\nu^2(t, \nu_t) - r \right) dt - \sigma_\nu(t, \nu_t) dW_{\nu t}. \tag{30}$$

We begin with some preliminary observations. As in Collin-Dufresne and Fos (2016), price impact is instantaneously negatively correlated with innovations in noise trading volatility. Moreover, the drift in $\lambda$ can either be positive or negative and generally depends on the current state of $\nu_t$. This is in contrast to earlier strategic trading models in which $\lambda$ is either constant (e.g., in Kyle (1985)), a martingale (e.g., in Back (1992)), a super-martingale (e.g., in Back and Baruch (2004)) or a sub-martingale (e.g., Collin-Dufresne and Fos (2016)). To gain some intuition, note that the equilibrium price impact must be such that the trader is indifferent between trading immediately or waiting, a result first established by Back (1992). On the one hand, stochastic volatility of noise trading generates an option to wait for higher liquidity in the future — the first term in the drift of $\lambda$ (i.e., $\nu^2\sigma_\nu^2(t,\nu)\frac{\gamma_\nu^2}{\gamma^2}$) tends to push price impact up over time to discourage waiting. On the other hand, the random terminal date induces early trading (since future periods are effectively discounted) — to offset this incentive, the second term in the drift (i.e., $-r$) pushes price impact lower on average. Which effect dominates depends on the relative size of the two effects and in general on the current state of noise trading volatility $\nu_t$.

More importantly, our analysis highlights that neither price impact nor return volatility are necessarily good measures of "adverse selection" when information acquisition is endogenous. Specifically, the optimal choice of precision $\eta_t = f\left(\frac{\Omega_t^2}{2\lambda_t}\right)$ is negatively related to price impact $\lambda_t$ — intuitively, the trader acquires more precise information when the market is more liquid. As such, shocks to the volatility of noise trading that lead to a decrease in price impact may simultaneously lead to more precise information being acquired, and consequently, an increase in adverse selection.[18] Similarly, note that when $\nu_t$ has a deterministic drift coefficient, instantaneous price variance is also deterministic since:

$$\sigma_P^2 = \lambda_t^2\nu_t^2 = \frac{e^{-2rt}\Sigma_t\nu_t^2}{G_t} = e^{-2rt}\frac{\Sigma_0 e^{-\int_0^t \frac{e^{-2rs}}{B(s)}ds}}{B\left(t\right)}, \tag{31}$$

while information acquisition evolves stochastically and, therefore, is unrelated to to price volatility.

These results are especially important for empirical analysis: our results imply that common proxies for privately informed trading (e.g., Kyle's $\lambda$ or return volatility) should be carefully interpreted in the presence of endogenous information acquisition. This is broadly consistent with Ben-Rephael et al. (2017) who document that firms for which institutional investors exhibit high abnormal attention also tend to have lower bid-ask spreads. On the other hand, our model implies that information acquisition is higher when trading activity (i.e., volatility of order flow, which is driven by noise trading volatility) is higher. This is consistent with the evidence of Ben-Rephael et al. (2017), who show that their measure of abnormal institutional investor attention is higher on days with higher abnormal trading

[18] This is similar to the observation made by Collin-Dufresne and Fos (2015), who show that Schedule 13D investors are more likely to *trade* on days when liquidity is high and empirical estimates of standard adverse selection measures are low. However, since their is no information acquisition in their model, the degree of information asymmetry falls over time (i.e., informational efficiency increases over time). In contrast, because the trader acquires information more aggressively when $\lambda$ is low, in our model the an increase in information asymmetry can coincide with a decrease in price impact, as we discuss below.

volume. Intuitively, this is because higher noise trading volatility increases the marginal value of acquiring private information.

Next, we turn to the question of how the informativeness of prices evolves in our model. Because our focus is on endogenous information acquisition, our analysis is well-suited to understanding the distinction between price informativeness and price efficiency. As emphasized by Weller (2017), *price informativeness* is an measure the *total* informational content of prices, while *informational efficiency* is a measure of how well prices aggregate *existing* private information. While the two concepts are closely related, they are distinct. For instance, if investors do not acquire a lot of information, but there is little noise trading, then prices may be very efficient, but not very informative.

A natural measure of price informativeness is

$$PI_t \equiv \frac{\Sigma_0 - \Sigma_t}{\Sigma_0}, \tag{32}$$

since it reflects the total amount of information that the market has learned about fundamentals. Similarly, a natural measure of informational efficiency is

$$IE_t \equiv \frac{\Omega_t}{\Sigma_t} \tag{33}$$

since it reflects ratio of the market maker's precision (i.e., $\Sigma_t^{-1}$) to the trader's precision (i.e., $\Omega_t^{-1}$). The next result characterizes how these measures evolve in our model.

**Proposition 1.** *The evolution of price informativeness, $PI_t$, is given by:*

$$dPI_t = -\frac{d\Sigma_t}{\Sigma_0} = e^{-2rt}\frac{\nu_t^2\Sigma_t}{G_t\Sigma_0}dt \tag{34}$$

*and the evolution of informational efficiency $IE_t$ is given by:*

$$dIE_t = \frac{d\Omega_t}{\Sigma_t} - \frac{d\Sigma_t}{\Sigma_t}IE_t, \text{ where } d\Omega_t = -\eta_t\Omega_t^2 dt, \text{ and } \eta_t = f\left(\frac{e^{rt}\Omega_t^2}{2}\sqrt{\frac{G_t}{\Sigma_t}}\right) \tag{35}$$

*Moreover, if $\nu_t$ has a deterministic drift coefficient, i.e., $\mu_\nu(t,\nu) = \mu_\nu(t)$, then (i) $\Sigma_t$ is deterministic, and (ii) information acquisition is given by $\eta_t = f\left(\frac{e^{rt}\nu_t\Omega_t^2}{2}\sqrt{\frac{B(t)}{\Sigma_t}}\right)$.*

The above result implies a number of novel empirical predictions.

First, somewhat surprisingly, price informativeness is unaffected by the cost of information acquisition. Intuitively, while the cost of information affects how quickly the trader acquires information, this does not affect her trading strategy because she optimally smoothes her use of information over time. Recall that the trader's equilibrium trading rate is

$$\beta_t(V_t - P_t) = \nu_t^2 \lambda_t \frac{V_t - P_t}{\Psi_t}$$

which is proportional to her informational advantage $V_t - P_t$ but scaled by the "size" of the advantage, measured in terms of the market-maker's conditional variance of $V_t$ (i.e., $\Psi_t$). When the market maker perceives a high information advantage (high $\Psi_t$) he is more sensitive to order flow; the trader responds by trading less aggressively. Similarly, when the perceived information advantage is low, prices are less sensitive to order flow, and the trader responds by trading more aggressively. We show that the equilibrium trading strategy is invariant to the rate of information acquisition. As a result, while the cost of information acquisition affects the rate of information acquisition by the trader, it does not affect the rate of learning by the market maker.[19] It is important to note that this invariance of price informativeness to information acquisition is a feature of information acquisition by a strategic trader and distinguishes our model from one in which investors trade competitively on their private information (e.g., Han (2018)).

Second, informational efficiency can increase with the cost of information acquisition. Note that a higher cost leads to a lower rate of information acquisition.[20] However, as just discussed, the market maker's rate of learning is invariant to the rate of information acquisition by the trader. As a result, the overall effect of an increase in information costs is to improve the *relative* informational position of the market maker versus the strategic trader.

Third, price informativeness and informational efficiency can move in opposite directions over time. Notably, price informativeness unambiguously increases over time, as the trader continuously trades on her information and thereby incorporates it into the price. However, price efficiency may increase or decrease over time. In particular, when the trader learns sufficiently quickly (i.e., $\eta_t \Omega_t^2$ is sufficiently large) relative to the market maker, informational efficiency falls. Moreover, this disconnect between informativeness and efficiency is more likely when noise trading volatility is high (i.e., $\nu_t$ is high) and when the trader's posterior uncertainty is high (i.e., $\Omega_t$ is high), since these lead to more intensive private information acquisition (i.e., higher signal precision).

Figure 1 illustrates these results when (i) noise trading volatility follows a geometric

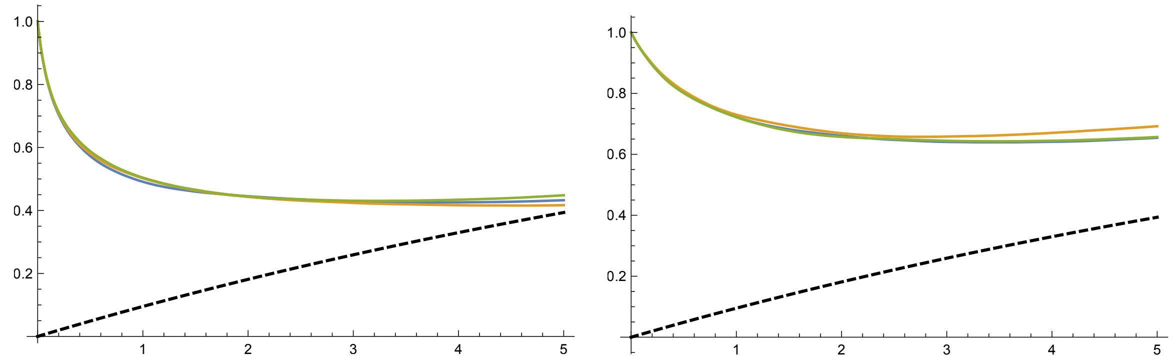
[19] This result is analogous to that of Back and Pedersen (1998) who establish that in continuous-time Kyle models with a fixed amount of Gaussian private information to be observed by the trader, the timing of information arrival is irrelevant for the trader's optimal trading strategy and market liquidity.

[20] Because $f = c^{-1}$ is the inverse cost function, an increase in the cost of information at all precisions appears as a similar decrease in $f$, which makes $d\Omega$ "less negative."

Brownian motion and (ii) information acquisition incurs a quadratic cost. As the above result implies, in this case, price informativeness is deterministic, while informational efficiency evolves stochastically (as it depends on the path of $\nu_t$). The dashed line in each panel plots the price informativeness and the solid lines information efficiency along various paths of $\nu_t$. As discussed above, informativeness increases over time while efficiency initially tends to decrease and then eventually increase. Moreover, while informativeness is unaffected by the cost of information acquisition, price efficiency is higher when the cost of information is higher (left panel).

Figure 1: Evolution of price informativeness $\Sigma_t$ and efficiency $\Psi_t$

Price informativeness $PI_t$ is plotted as the dashed line, while information efficiency $IE_t$ paths (for different paths of $\nu_t$) are plotted as solid lines. The cost of information acquisition is given by $c(\eta) = \frac{c_0}{2}\eta^2$. Other parameters are set to $\Sigma_0 = 1$, $r = 0.05$, $\mu_\nu = 0$, $\sigma_\nu = 0.1$, and $\nu_0 = 1$.




(a) low cost (i.e., $c_0 = 0.5$) (b) high cost (i.e., $c_0 = 2$)

Taken together our results suggest that endogenous information acquisition generates novel predictions and empirical implications that distinguish it from standard strategic-trading settings with exogenous information endowment. For instance, much of the literature has interpreted high price impact (high $\lambda$) as evidence of more asymmetric information, consistent with models in which strategic traders are exogenously endowed with information. However, we show more information is acquired when price impact is low, which tends to increase information asymmetry. Similarly, in most existing models of strategic trading, price informativeness and efficiency both unambiguously increase over time. In contrast, we show that when the investor acquires information more quickly than it is incorporated into prices, endogenous information acquisition can lead to a decrease in efficiency even though informativeness increases over time. As such, one must be careful in applying the intuition developed in standard, strategic trading models to settings in which endogenous, "smooth" information acquisition plays an important role.

# 4 Lumpy Acquisition

In this section, we study settings in which the information acquisition technology is "lumpy" in that there is a fixed cost involved with information collection (i.e., the cost of information, starting from *zero* precision, includes a non-zero "lump" component). Specifically, we consider a setting where the trader acquires a signal $S$ at any stopping time $\tau \in \mathcal{T}$ by paying a cost $c > 0$, where $\mathcal{T}$ denotes the set of $\mathcal{F}_t^P$ stopping times.[21] As discussed earlier, we assume that (before acquiring any information), the trader's acquisition depends only on public information up to that point. We allow for both pure and mixed acquisition strategies by assuming the trader's strategy is a probability distribution over stopping times $\tau \in \mathcal{T}$.[22] Importantly, such strategies can involve both "continuous" mixing in which the trader acquires information with a given intensity over an interval of times, as well as "discrete" mixing in which the trader acquires at a countable collection of times. If the probability distribution over $\mathcal{T}$ is degenerate, then the equilibrium is one of pure-strategy acquisition.

We show that, in contrast to the results from the previous section, there do not exist equilibria with endogenous information acquisition in this case. Section 4.1 highlights the key economic forces that give rise to non-existence of equilibria. First, we show there cannot exist pure strategy equilibria in which acquisition occurs with delay, since the trader can deviate by preemptively acquiring information earlier and trading against an unresponsive pricing rule. Second, in any setting with "trade timing indifference," (defined below) the strategic trader can profitably deviate by waiting — while her expected trading gains are unaffected due to the indifference condition, she benefits by delaying the cost of acquisition. This rules out the existence of both pure strategy equilibria with immediate acquisition and mixed strategy equilibria. The subsection ends with an informal discussion of the conditions that naturally give rise trade timing indifference in our setting.[23]

Section 4.2 explores the robustness of our non-existence result to alternate assumptions. We consider settings in which (i) the strategic trader is risk-averse, (ii) the market maker is not competitive, (iii) the informed investor is initially informed, (iv) trading occurs suf-

[21] The assumption that the trader observes a a single, though possibly multidimensional, "lump" of information is not crucial for our results. What is crucial is that obtaining information involves paying a fixed cost. As such, our results are robust to information technologies that provide the trader with a future flow of signals in return for an upfront cost of $c$. We discuss this in more detail in Section 4.2.

[22] That is, at the beginning of the game, the trader randomly chooses a stopping time according to some probability distribution, and follows the realized strategy for the duration of the game. There are multiple, equivalent ways of defining randomization over stopping times. Aumann (1964) introduced the notion of randomizing by choosing a stopping time according to some probability distribution at the start of the game. Touzi and Vieille (2002) treat randomization by identifying the stopping strategy with an adapted, non-decreasing, right-continuous processes on $[0, 1]$ that represents the cdf of the time that stopping occurs. Shmaya and Solan (2014) show, under weak conditions, that these definitions are equivalent.

[23] The formal results are presented in Appendix B.

ficiently frequently, but not continuously, and (v) there is no discounting. We show that non-existence of equilibria is a robust outcome with lumpy information acquisition in all of these settings.

## 4.1 Key Economic Forces

This sub-section presents the main economic forces that drive non-existence of equilibria with lumpy information costs. Our first observation is immediate: pure strategy equilibria in which information is acquired with some delay cannot be an equilibrium.

**Proposition 2.** *(Pre-emption Deviation) There does not exist an equilibrium in which the trader follows a pure acquisition strategy that acquires information after time $t = 0$ with positive probability. That is, there does not exist any equilibrium in pure strategies in which there is a time $s > 0$ such that $\mathbb{P}(\tau > s) > 0$.*

*Proof.* Suppose that there does exist such an equilibrium. Then in equilibrium, the order flow is completely uninformative about $V$ prior to time $\tau$ and therefore the pricing rule is insensitive to order flow before $\tau$. But in the event $\{\tau > 0\}$ (which occurs with strictly positive probability), the strategic trader can profitably deviate by unobservable acquiring information prior to $\tau$ and trading at an arbitrarily large rate with zero price impact, thereby generating unbounded profits. Since acquisition is unobservable by the market maker, he cannot respond to the deviation by adjusting the price impact. □

Intuitively, the result follows from the fact that when acquisition cannot be detected, the strategic trader cannot commit to acquiring information at a future date: she always finds it profitable to deviate by pre-empting herself and acquiring information earlier. The lack of commitment leads to non-existence of pure-strategy equilibria with delay in information acquisition. It also immediately rules out equilibria in which information is never acquired. Moreover, this incentive to preemptively acquire information is likely to apply more generally, e.g., in settings with multiple investors, in discrete time, and in settings with a fixed terminal date.

The above result implies that with unobservable acquisition, the only remaining candidates for a equilibrium are those in which (i) the trader follows a pure acquisition strategy that acquires immediately, $\mathbb{P}(\tau = 0) = 1$, or (ii) the trader follows a mixed acquisition strategy. Note that in a non-degenerate mixed strategy equilibrium, any stopping time $\tau$ in the set of stopping times on which the trader's acquisition strategy places positive probability must have strictly positive probability of acquisition in *any* neighborhood of zero (i.e., for *any* $\Delta > 0$, $\mathbb{P}(\tau \in (0, \Delta)) > 0$)). If any such stopping times did not satisfy this, then, conditional on that stopping time being realized from the original mixing randomization, the trader could profitably deviate by preempting the conjectured strategy as in Proposition 2. Because both of the remaining candidate equilibria have $\mathbb{P}(\tau \in [0, \Delta)) > 0$), if we can rule out equilibria with such a property, then we have eliminated all candidate equilibria. Our next result establishes that if, in equilibrium, an informed trader's problem exhibits **"trade timing indifference**," then we can, in fact, rule out such equilibria.

Before doing so, we precisely define trade timing indifference:

**Definition 2.** An equilibrium features ***trade timing indifference*** if at any date $t$ and for each $\Delta > 0$, the change in expected profit for an informed trader over the interval $[t, t + \Delta)$, if she does not trade over this interval and follows her conjectured equilibrium strategy afterwards, is zero. i.e., if $dX_t^s = 0$, $t \in [t, (t + \Delta))$ implies[24]

$$\mathbb{E}_t^s \left[ e^{-r\Delta} J^s \left( (t + \Delta)^-, \cdot \right) - J^s \left( t^-, \cdot \right) \right] = 0. \tag{36}$$

As first noted by Back (1992), the above is a key feature of continuous-time Kyle models: over any finite interval of time, the trader is indifferent between playing her equilibrium trading strategy or refraining from trade over that interval and then trading optimally from that time forward. Economically, this result arises because an equilibrium pricing rule must be such that the trader does not perceive any exploitable predictability in the price level or price impact if she refrains from trading (i.e., any predictability that can be profitably exploited, accounting for the random horizon). Otherwise, she would have a profitable deviation from her conjectured equilibrium *trading* strategy. As we shall discuss below, it arises naturally in any (Markovian) equilibrium of our model. Moreover, as we highlight in Section 4.2, it also arises in other related settings.

The next result establishes the non-existence of equilibria that feature trade timing indifference and endogenous information acquisition.

**Proposition 3.** *(Delay Deviation) Fix any $\Delta > 0$. There does not exist an equilibrium in which (i) trade timing indifference holds, and (ii) the strategic trader acquires information with positive probability in $[0, \Delta)$, i.e., in which $\mathbb{P}(\tau \in [0, \Delta)) > 0$).*

*Proof.* Suppose there is an equilibrium and let $\tau \in \mathcal{T}$ be the trader's acquisition strategy. Let $\bar{J}(t, \cdot)$ denote the gross expected profit from acquiring information as of time $t$ given

[24]There are two points in this definition that are worth clarifying. First, the $^-$ signs in (36) arise because we have not restricted the trader to smooth strategies. In principle, she could acquire at time $t$ and then immediately submit a discrete order. Because $J(t, \cdot)$ represents her forward-looking expected profit, we must evaluate at $t^-$ to capture any discrete trade at $t$ in the value function. Secondly, strictly speaking we require (36) to hold at any $\left\{\mathcal{F}_t^P\right\}$ stopping time $\tau$. Of course, since $\tau = t$ is a well defined stopping time for any given $t$, this implies (36).

that one has not acquired information previously

$$\bar{J}(t,\cdot) = \mathbb{E}\left[J^S(t,\cdot)\,\middle|\mathcal{F}_t^P\right].$$

Notice that we must have $J^U(\tau^-,\cdot) \leq \bar{J}(\tau^-,\cdot) - c$, where $J^U(t,\cdot)$ denotes the value function of the uninformed trader. Consider the following deviation by the trader: do not acquire information at $\tau$, do not trade in $[\tau, \tau+\Delta)$, and then acquire at $t = \tau + \Delta$ and follow the conjectured equilibrium trading strategy from that point forward. The expected profit from this deviation is

$$\begin{aligned}
\bar{\Pi}_{d\tau} &\equiv e^{-r\Delta}\mathbb{E}_\tau[\bar{J}((\tau+\Delta)^-,\cdot) - c] - J^U(\tau^-,\cdot) \\
&\geq e^{-r\Delta}\mathbb{E}_\tau[\bar{J}((\tau+\Delta)^-,\cdot) - c] - (\bar{J}(\tau^-,\cdot) - c) && (37)\\
&= (1-e^{-r\Delta})c + \mathbb{E}_\tau[e^{-r\Delta}\bar{J}((\tau+\Delta)^-,\cdot) - \bar{J}(\tau^-,\cdot)] && (38)\\
&= \left(1-e^{-r\Delta}\right)c > 0, && (39)
\end{aligned}$$

where the final equality follows from the observation that trade timing indifference implies $\mathbb{E}_\tau\left[\mathbb{E}_\tau^s\left[e^{-r\Delta}J^s((\tau+\Delta)^-,\cdot) - J^s(\tau^-,\cdot)\right]\right] = 0$. □

Intuitively, the delay deviation is as follows: instead of acquiring information at time $\tau$, the strategic trader can instead *wait* over the interval $[\tau, \tau+\Delta)$, during which she does not acquire and does not trade, and then acquire information.[25] Given that future periods are discounted (due to the stochastic horizon $T$), she benefits from delaying the cost of acquisition, but forgoes trading gains. Due to trade timing indifference, the expected loss in trading gains is zero. However, since discounted trading costs are of order $\Delta$, the deviation leaves the trader strictly better off.

To summarize, one can rule out existence of equilibria with endogenous information acquisition if trade timing indifference holds. In Appendix B, we formally characterize sufficient conditions under which any Markovian equilibrium of our model must feature trade timing indifference — we informally outline the steps here. Suppose the risky asset price $P_t$ is a (sufficiently smooth) function of stochastic noise trading volatility $\nu$ and a (finite, but arbitrarily large) set of Markovian state variables $p$, with evolution

$$dp = \mu_p(t,p,\nu)\,dt + \sigma_p(t,p,\nu)\,dW_{pt} + \mathbf{1}dY, \tag{40}$$

which summarize the market maker's beliefs about trader's private information $S$.[26] First, we

[25] Notably, the deviation holds at all times of conjectured acquisition, even those outside of $[0,\Delta)$.

[26] As we discuss in the Appendix, the coefficient on order-flow is normalized to 1 without loss of generality

assume that there exists a solution to each informed trader type's Hamilton-Jacobi-Bellman (HJB) equation:

$$0 = \sup_{\theta} \left\{ \begin{array}{c} -rJ^s + J^s_t + J^s_\nu \cdot \mu_\nu + J^s_p \cdot (\mu_p + \mathbf{1}\theta) + \theta\,(V - P_{t^-}) \\ +\frac{1}{2}\sigma^2_\nu J^s_{\nu\nu} + \frac{1}{2}\mathrm{tr}\left(J^s_{pp}(\sigma_p\sigma_p' + \nu^2\mathbf{1}\mathbf{1}')\right) + \mathrm{tr}\left(J^s_{\nu p}\sigma_p\sigma_\nu\mathbf{1}'\right) \end{array} \right\}. \tag{41}$$

We formalize this assumption in Assumption (67) in the Appendix. It is important to note that we do *not* assume that the HJB equation characterizes the value function, nor that the trader finds it optimal to trade smoothly, but merely that there exists a sufficiently smooth function that satisfies the HJB equation.

Second, Proposition 5 establishes that in any equilibrium, if it were to exist, an informed trader's optimal trading strategy is absolutely continuous i.e., $dX_t = \theta(\cdot)\,dt$, where $\theta(\cdot)$ denotes the trading rate and her value function is, in fact, characterized by the HJB equation. The optimality of a smooth trading strategy extends the arguments in Kyle (1985) and Back (1992) to our setting, and arises in all continuous-time, strategic trading models we are aware of. Intuitively, if an informed trader does not trade smoothly she reveals her information too quickly, and this is not optimal. Furthermore, because eq. (41) is linear in $\theta$, and $\theta$ is unconstrained, it follows that the sum of the coefficients on $\theta$ must be identically zero and therefore the sum of the remaining terms must also equal zero i.e.,

$$\begin{array}{c} -rJ^s + J^s_t + J^s_\nu \cdot \mu_\nu + J^s_p \cdot \mu_p \\ +\frac{1}{2}\sigma^2_\nu J^s_{\nu\nu} + \frac{1}{2}\mathrm{tr}\left(J^s_{pp}(\sigma_p\sigma_p' + \mathbf{1}\nu^2\mathbf{1}')\right) + \mathrm{tr}\left(J^s_{\nu p}\sigma_p\sigma_\nu\mathbf{1}'\right) \end{array} = 0 \tag{42}$$

But the above is simply the expected differential of the value function of an informed investor under the assumption that her trading rate at $t$ is zero i.e., $\theta_t = 0$. This establishes trade timing indifference and we have the following result.

**Theorem 2.** *There does not exist an equilibrium in which the trader follows a pure information acquisition strategy in which information is acquired after $t = 0$ with positive probability. Moreover, if Assumption 1 holds, there does not exist any equilibrium with costly information acquisition.*

The above results rule out the existence of equilibrium in the case of unobservable information acquisition, under standard regularity conditions. The deviation arguments apply generally to a large class of models that feature discounting (e.g., Back and Baruch (2004), Chau and Vayanos (2008), Caldentey and Stacchetti (2010)), which implies that the trading equilibria in these models do not naturally arise as a consequence of costly dynamic

— since $P = g(t, \nu, p)$ for a function $g(\cdot)$, "Kyle's lambda" is given by $g_p \cdot \mathbf{1}$. We assume that $g$ is continuously differentiable in $t$ and twice continuously differentiable in the state variables $\{\nu, p\}$.

information acquisition. While these models provide useful intuition for how exogenous (or costless), private information gets incorporated into prices, our analysis recommends caution when considering settings with lumpy, endogenous information acquisition. Moreover, as we discuss next, our results are qualitatively similar in related settings.

## 4.2 Robustness

Pre-emption and delay are both robust, economically important forces that arise in dynamic settings. This section explores on the robustness of the delay deviation to alternate settings, since establishing the existence of such a deviation is sufficient to rule out both pure and mixed strategy equilibria.

### 4.2.1 Preferences of the market maker

A possible concern about our non-existence results is that they are a consequence of the assumption that the market maker is perfectly competitive and set the price as described in (3). Recent work in related models suggests that alternate assumptions about the market maker (e.g., that he is a profit maximizing monopolist) may help restore existence.[27] However, this does not recover existence of equilibria in our setting. Importantly, the arguments in Propositions 2 and 3 do not rely on whether the market maker sets prices competitively. Similarly, Assumption 1 and consequently Proposition 5 in Appendix B apply broadly to settings in which the price is a sufficiently smooth function of the underlying state variables. As such, the pre-emption and delay deviations apply to a larger class of models in which the market maker is not necessarily risk-neutral nor competitive. The key feature that is required for our delay argument is trade timing indifference, which arises as long as price changes and price-impact changes are not predictable when the informed strategic trader does not trade.

### 4.2.2 Risk aversion of the strategic trader

In this subsection, we explore the effect of allowing the strategic trader to be risk-averse. Suppose that a trader with $S = s$ has utility $u\left(T, w_T^s\right)$ over her terminal wealth $w_T^s$, where

$$w_t^s = \int_0^t \left(V - P_u\right) dX_u^s, \tag{43}$$

[27]For instance, Dai et al. (2019) establish non-existence of equilibria when a risk-neutral, competitive market maker faces uncertainty about the existence of a strategic trader, but show that existence can be restored when the market maker is monopolistic.

and let her continuation value function be denoted by[28]

$$J^s\left(t, w_t, \nu_t, p_t\right) = \sup_{X^s} \mathbb{E}_t\left[\int_t^T u\left(\ell, w_\ell\right) d\ell\right] = \sup_{X^s} \mathbb{E}_t\left[\int_t^\infty e^{-r(\ell-t)} u\left(\ell, w_\ell\right) d\ell\right], \tag{44}$$

conditional on the economy not having ended as of $t$. We argue that the delay deviation of 3 also applies since the equilibrium must feature trade timing indifference.

Arguments analogous to Proposition 5 imply that in equilibrium, any optimal trading strategy for an informed trader is absolutely continuous (i.e., $dX_t^s = \theta_t^s dt$), and the value function above satisfies the following HJB equation:

$$0 = \sup_\theta \left\{ \begin{array}{c} r\left(u\left(t, w_t\right) - J^s\right) + J_t^s + J_\nu^s \cdot \mu + J_p^s \cdot \left(\alpha + \mathbf{1}\theta\right) + \theta\left(V_t - P_{t^-}\right) J_w \\ + \frac{1}{2}\sigma_\nu^2 J_{\nu\nu}^s + \frac{1}{2}\mathrm{tr}\left(J_{pp}^s(\sigma_p\sigma_p' + \nu^2\mathbf{1}\mathbf{1}')\right) + \mathrm{tr}\left(J_{\nu p}^s \sigma_p \sigma_\nu \mathbf{1}'\right) \end{array} \right\}, \tag{45}$$

As before, the above problem is linear in $\theta$ and so a finite optimum requires:

$$J_p^s \cdot \mathbf{1} + \left(V_t - P_{t^-}\right) J_w = 0, \text{ and} \tag{46}$$

$$\left\{ \begin{array}{c} r\left(u\left(t, w_t\right) - J^s\right) + J_t^s + J_\nu^s \cdot \mu + J_p^s \cdot \alpha \\ + \frac{1}{2}\sigma_\nu^2 J_{\nu\nu}^s + \frac{1}{2}\mathrm{tr}\left(J_{pp}^s(\sigma_p\sigma_p' + \nu^2\mathbf{1}\mathbf{1}')\right) + \mathrm{tr}\left(J_{\nu p}^s \sigma_p \sigma_\nu \mathbf{1}'\right) \end{array} \right\} = 0. \tag{47}$$

But the latter equation reflects the change in expected utility over an instant $dt$, accounting for the possibility that the economy ends with probability $r\,dt$ (in which case, the change is $u - J^s$ because the trader gets to consume her terminal wealth but loses future profitable trading opportunities). Because the change in expected utility, given no trade, is zero, this implies that the trader must be indifferent between her posited optimal strategy and not trading over an interval and then following her optimal strategy i.e., that the equilibrium must feature trade timing indifference. As before, this implies there cannot be an equilibrium with unobservable information acquisition, even when the strategic trader is risk-averse.

### 4.2.3 Initial information endowment

Note that our nonexistence results largely survive if the trader is initially endowed with some private information about the payoff and can choose to pay cost $c > 0$ to observe some additional piece(s) of information at a time of her choosing. In this case, it is easy to show that as long as Assumption 1 holds, then such a trader always has a profitable

[28]Notice that there is a subtle difference with the value functions that we considered in the risk-neutral case above. In that case, we defined the value function as the expected trading profits from time $t$ onward, rather than the expected terminal wealth. When the trader is risk-neutral this distinction is immaterial since maximizing future trading profits and maximizing terminal wealth are equivalent. However, if the trader is not risk-neutral, we must explicitly keep track of her expected utility over the level of terminal wealth itself.

"delay deviation" from any conjectured information acquisition strategy. Under Assumption 1 this once again rules out the existence of any equilibria (pure or mixed strategy) with costly information acquisition. Even in the absence of Assumption 1, we suspect, but have not been able to prove, that a pre-emption deviation argument analogous to the one we considered above also rules out pure strategy equilibria with acquisition after $t = 0$.

#### 4.2.4 Discrete time

Next, we consider unobservable information acquisition in the discrete-time model of Caldentey and Stacchetti (2010). As before, Proposition 2 applies: never acquiring information, or acquiring it with a delay, cannot be an equilibrium. In Section C.1 of the Appendix, we show that information acquisition at date zero is not an equilibrium when the length between trading rounds is sufficiently small. Even though trade timing indifference does not arise in a discrete time model, the argument follows that of Proposition 3: instead of acquiring immediately, the strategic trader can wait for a period of length $\Delta$ and re-evaluate her decision. The expected gain from delaying acquisition is of order $\Delta$, but the expected loss from not trading in the first period is of order smaller than $\Delta$. As we show in Proposition 6, this implies that when $\Delta$ is sufficiently small, the deviation is strictly profitable.

The results from this analysis suggests that the conclusions of Section 4.1 do not rely on the assumption of continuous trading. In particular, when the time between trading dates is sufficiently small, the trading equilibrium in the discrete time setting of Caldentey and Stacchetti (2010) cannot arise endogenously as an outcome of unobservable, costly information acquisition. Given the compelling economic forces behind the result, we conjecture, but have not been able to prove, that similar arguments rule out mixed strategy acquisition when the time between trading rounds is sufficiently small. Note that, as in all of the above results, a positive discount rate (which is induced by the random horizon, though could be introduced explicitly) plays a crucial role in non-existence of equilibrium: the profitable deviation arises because by delaying acquisition, the present value of the information cost is (strictly) lower. However, as we show next, the delay deviation may also restrict the existence of equilibria even in settings without discounting.

#### 4.2.5 No discounting

In this subsection, we study unobservable information acquisition in the continuous-time version of Kyle (1985). Trading takes place on the interval $[0, 1]$. An identical argument to that in Proposition 2 immediately implies that any pure-strategy equilibrium cannot involve acquisition after time zero. Now, suppose that there is a pure strategy equilibrium in which the trader acquires information immediately at $t = 0$. In such an equilibrium, the pricing rule and the trader's post-acquisition value function (i.e., $J(t, y)$) are those from Kyle (1985) (or the special case of Back (1992) with normally-distributed payoff). Hence, $P(t, y) = \lambda y$ where $\lambda = \sqrt{\frac{\Sigma_0}{\sigma_Z^2}}$ and the ex-ante (gross) expected profit from being informed is

$$\overline{J}(0, 0) = \mathbb{E}[J^V(0, 0)] = \sqrt{\Sigma_0 \sigma_Z}$$

We would like to compare the above to the expected payoff if the trader deviates, remains uninformed for the duration of the trading game, and trades against the posited equilibrium price function. Following the argument of Proposition 1 in Back (1992), it is straightforward to construct the trader's value function under this deviation. It is

$$J^{d,U}(t, y) = J^0(t, y) = \frac{1}{2}\sqrt{\frac{\Sigma_0}{\sigma_Z^2}}y^2 + \frac{1}{2}\sqrt{\Sigma_0 \sigma_Z^2}(1 - t).$$

At time zero, this becomes

$$J^{d,U}(0, 0) = \frac{1}{2}\sqrt{\Sigma_0 \sigma_Z^2},$$

which is half of the informed trader's ex-ante gross profit. The fact that this trading profit is not zero is a consequence of the fact that in a dynamic model the trader expects profitable trading opportunities to arise in the future when the realized noise trade pushes the price away from zero. This implies the following result.

**Proposition 4.** *Suppose that Assumption 1 holds, and $c > \frac{1}{2}\sqrt{\Sigma_0 \sigma_Z^2}$. Then, there does not exist an equilibrium in which information acquisition follows a pure strategy.*

The above follows from the observation that when the cost is sufficiently high, the net expected profit from deviating and never acquiring information is strictly positive i.e.,

$$J^{d,U}(0, 0) - (\bar{J}(0, 0) - c) = c - \frac{1}{2}\sqrt{\Sigma_0 \sigma_Z^2} > 0, \tag{48}$$

but this implies the trader never acquires — which cannot be an equilibrium. Importantly, note that the above result implies that when $c > \frac{1}{2}\sqrt{\Sigma_0 \sigma_Z^2}$, the financial market equilibrium in Kyle (1985) cannot arise as a consequence of endogenous information acquisition. This result is a stronger version of the delay deviation in Proposition 3: when the cost of information is sufficiently high, it is profitable for the trader to deviate by *never* acquiring information.[29]

[29] In fact, nonexistence holds even if we force the trader to make her acquisition decision at $t = 0$, as nothing in the proof above relied on deviating on intermediate dates.

# 5 Concluding remarks

We introduce dynamic, endogenous information acquisition in an otherwise standard strategic trading environment, by allowing the trader to choose *when* to acquire information about the asset payoff. We consider two types of information acquisition technology. When the strategic trader can choose the precision of a flow of private information subject to a smooth cost of precision, we show that she optimally chooses higher precision when uninformed trading volatility and market liquidity are high. Moreover, the equilibrium with endogenous information acquisition has qualitatively different implications for the dynamics of liquidity and informational efficiency than the standard equilibrium with endowed exogenous information. On the other hand, when the cost of information is lumpy, we show that the equilibrium breaks down: we show there cannot exist with strategic trading and information acquisition (in either pure or mixed strategies) when a standard "trade-timing indifference" result holds (e.g., in any Markovian equilibrium). Intuitively, in this case, we show that the trader can always profitably deviate by either (i) pre-emptively acquiring information earlier than expected, or (ii) delaying acquisition past the prescribed time.

As highlighted in our analysis, the nature of equilibrium is qualitatively different with endogenous information acquisition versus without. This is particularly important for empirical and policy analysis, as insights about market liquidity and the informational efficiency of prices change qualitatively once we introduce information choice. Our analysis suggests a number of avenues for future research.

**Information acquisition technologies.** As highlighted by the stark difference in conclusions between Sections 3 and 4, the specific nature of the information acquisition technology can have very important consequences. It would be interesting to explore (i) the robustness of the results, and (ii) the implications on liquidity and informational efficiency for a larger class of acquisition technologies.

**Endowed and acquired information.** The focus of this paper is to highlight the implications of endogenous information acquisition, and as such, we assume the trader is not endowed with any private information. As we discuss in Section 2.1, our results are qualitatively similar if the trader is endowed with payoff relevant information. However, it would be interesting to explore settings where the trader is endowed with private information about preferences / costs that affect her information acquisition choices. For instance, if the trader is privately informed about her cost of information acquisition, are market uncertainty and price impact invariant to her information acquisition choices?

**Detectability of information acquisition**. Market breakdown arises because the strategic trader can deviate in her acquisition strategy without being detected. In settings where acquisition is detectable by other market participants, these deviations may no longer be profitable. For instance, Banerjee and Breon-Drish (2018) consider a setting where entry into new markets is observable, and show that strategic trading equilibria can be sustained. They show that allowing for flexibility in timing of information acquisition and entry leads to novel predictions for the likelihood of informed trading, entry dynamics and optimal precision choice.

**Trading Frictions.** Our analysis suggests that accounting for trading frictions (e.g., restrictions on the trading rate, or costs of trading faster) might be another important aspect of understanding information acquisition in strategic trading environments. In particular, introducing trading frictions in a manner that eliminates trade timing indifference may restore the existence of equilibria.

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# A Proofs of Results from Section 3

## A.1 Proof of Proposition 1.

The goal of this section is to establish the result in Proposition 1. The proof largely replicates that in Collin-Dufresne and Fos (2016), modified as appropriate for the endogeneity of trader beliefs. Recall the conjectures in the text:

$$dP = \lambda_t dY$$
$$\theta = \beta_t (V_t - P_t)$$
$$\chi = \eta_t$$

for $\mathcal{F}_t^P$-adapted processes $\lambda_t, \beta_t, \eta_t$, and a strategic trader value function of the form

$$J = \frac{(V_t - P_t)^2 + \kappa_{1t}}{2\lambda_t} + \kappa_{2t}$$

for some locally-deterministic stochastic processes $\kappa_{jt}$ to be determined.

We need to show that (i) given the conjectured trading and information acquisition strategy, the optimal pricing rule takes the conjectured form, and (ii) given the conjectured pricing rule, the optimal trading and information acquisition strategy take the conjectured form. We proceed by first considering the market maker's filtering problem, then the trader's optimization problem, and then jointly enforcing the optimality conditions derived for each agent.

### A.1.1 The market maker's problem

The market maker must filter the asset value from the order flow. Because the trader is better informed than the market maker, the law of iterated expectations implies that it is sufficient that he filter the trader's value estimate from order flow.

Given an arbitrary $\mathcal{F}_t^P$-adapted precision process $\eta_t$, standard filtering arguments (e.g., Liptser and Shiryaev (2001)) deliver the dynamic's of the trader's conditional mean and variance as

$$dV_t = \sqrt{\eta_t}\Omega_t(ds - v_t dt) = \sqrt{\eta_t}\Omega_t d\widehat{W}_{vt}$$
$$d\Omega = -\eta_t \Omega_t^2 dt$$

where $\widehat{W}_{vt}$ is a Brownian motion under the trader's filtration.

The market maker's objective is to compute $\mathbb{E}[V_t|\mathcal{F}_t^P]$ from observing

$$dY = dX + dZ = \beta_t(V_t - P_t)dt + \nu_t dW_{Zt}$$

Under the conjecture that the precision process is $\mathcal{F}_t^P$-adapted, we can once again apply standard filtering theory to conclude that the market's maker's conditional mean and variance, $(\overline{v}, \Psi)$, are characterized by

$$\begin{aligned} d\bar{V}_t &= \frac{\beta_t \Psi_t}{\nu_t^2}\left(dY_t - \beta_t(\bar{V}_t - P_t)dt\right) \\ d\Psi &= \eta_t \Omega_t^2 - \frac{\beta_t^2 \Psi_t^2}{\nu_t^2} dt. \end{aligned}$$

Enforcing efficient pricing $P_t = \overline{V}_t$ and then setting coefficients equal to the conjectured optimal pricing rule yields

$$\lambda_t = \frac{\beta_t \Psi_t}{\nu_t^2} \tag{49}$$

with

$$d\Psi_t = \eta_t \Omega_t^2 - \nu_t^2 \lambda_t^2. \tag{50}$$

### A.1.2 The trader's problem

We will use the HJB equation to determine the optimal strategy and value function coefficients.[30] Let $M_t = V_t - P_t$ and write the trader's conjectured value function as

$$J = \frac{M_t^2 + \kappa_t}{2\lambda_t}.$$

From above, the trader's conditional mean and variance, under arbitrary precision process $\eta_t$ are

$$\begin{aligned} dV_t &= \sqrt{\eta_t}\Omega_t(ds - V_t dt) = \sqrt{\eta_t}\Omega_t d\widehat{W}_{Vt} \\ d\Omega_t &= -\eta_t \Omega_t^2 dt \end{aligned}$$

[30]It is straightforward to use standard verification arguments (see, e.g, Caldentey and Stacchetti (2010)) to show that the resulting strategies are in fact optimal and that the given function $J$ does, in fact, characterize the trader's optimum.

where $\widehat{W}_{Vt}$ is an independent Brownian motion under the trader's filtration. Under the conjectured pricing rule, $M_t$ therefore follows

$$
\begin{aligned}
dM_t &= dV_t - dP_t \\
&= -\lambda_t dY_t + \sqrt{\eta_t}\Omega_t d\widehat{W}_{Vt} \\
&- \lambda_t \theta dt - \lambda_t \nu_t dW_{Zt} + \sqrt{\eta_t}\Omega_t d\widehat{W}_{Vt}.
\end{aligned}
$$

Suppressing the arguments of functions, the trader's HJB equation is

$$
0 = \sup_{\theta\in\mathbb{R},\eta>0} \mathbb{E}_t \left[ \begin{array}{c} -rJdt + J_{\kappa_1} d\kappa_{1t} + J_{\kappa_2} d\kappa_{2t} + J_M dM_t + J_{(1/\lambda)} d\left(1/\lambda_t\right) \\ +\frac{1}{2} J_{MM} \left(dM_t\right)^2 + J_{M(1/\lambda)} dM_t d\left(1/\lambda\right)_t + \theta M_t dt \\ -c\left(\eta\right) dt \end{array} \right] \tag{51}
$$

$$
= \sup_{\theta\in\mathbb{R},\eta>0} \mathbb{E}_t \left[ \begin{array}{c} -r\frac{M_t^2+\kappa_{1t}}{2\lambda_t} dt - r\kappa_{2t} dt + \frac{1}{2\lambda_t} d\kappa_{1t} + d\kappa_{2t} + \frac{M_t}{\lambda_t} dM_t + \frac{M_t^2+\kappa_t}{2} d\left(1/\lambda_t\right) \\ +\frac{1}{2}\frac{1}{\lambda_t} \left(dM_t\right)^2 + M_t dM_t d\left(1/\lambda\right)_t + \theta M_t dt \\ -c\left(\eta\right) dt \end{array} \right] \tag{52}
$$

The first-order conditions with respect to $\theta$ and $\chi$ require

$$
\begin{aligned}
-J_M \lambda + M &= 0 \\
\frac{1}{2} J_{MM} \Omega^2 - c'(\eta) &= 0.
\end{aligned}
$$

Note that because of the convexity of $c(\cdot)$, the second-order condition that the Hessian with respect to $(\theta, \eta)$ is positive semi-definite is automatically satisfied. Using the conjectured value function and inverting the marginal cost $(c')^{-1} = f$ to solve explicitly for the precision (which is permissible because $c'' > 0$ and $c'$ is surjective onto $[0, \infty)$) yields

$$
-\frac{1}{\lambda_t} M \lambda_t + M = 0 \tag{53}
$$

$$
\eta_t^* = f\left(\frac{1}{2\lambda_t}\Omega_t^2\right). \tag{54}
$$

The $\theta$ FOC is satisfied trivially, and plugging the optimal $\eta_t$ (denoted by a $*$) back into the

HJB equation gives

$$\begin{aligned}
0 &= \mathbb{E}_t \left[ \begin{array}{c} -r\frac{M_t^2+\kappa_{1t}}{2\lambda_t}dt - r\kappa_{2t}dt + \frac{1}{2\lambda_t}d\kappa_{1t} + d\kappa_{2t} + \frac{M_t^2+\kappa_{1t}}{2}d\left(1/\lambda_t\right) \\ +\frac{1}{2\lambda_t}\left(dM_t\right)^2 + M_t dM_t d\left(1/\lambda\right)_t \\ -c\left(\eta_t^*\right)dt \end{array} \right] \\
&= \begin{array}{c} \frac{\left(M_t^2+\kappa_{1t}\right)}{2}\left(E_t\left[d\left(1/\lambda_t\right)\right] - \frac{r}{\lambda_t}dt\right) + \frac{1}{2\lambda_t}\left(d\kappa_{1t} + \left(\eta_t^*\Omega_t^2 + \lambda_t^2\nu_t^2\right)dt\right) \\ + \left(d\kappa_{2t} - \left(r\kappa_{2t} + c\left(\eta_t^*\right)\right)\right) + M_t E_t\left[dM_t d\left(1/\lambda\right)_t\right] \end{array}
\end{aligned}$$

For this expression to equal zero at all points in the state space, we require

$$E_t\left[d\left(1/\lambda_t\right)\right] = \frac{r}{\lambda_t}dt$$

with $E_t\left[dM_t d\left(1/\lambda\right)_t\right] = 0$, $d\kappa_{1t}$ and $d\kappa_{2t}$ be defined by

$$\begin{aligned}
d\kappa_{1t} &= -\left(\eta_t^*\Omega_t^2 + \lambda_t^2\nu_t^2\right)dt \\
d\kappa_{2t} &= \left(r\kappa_{2t} + c(\eta_t^*)\right)dt
\end{aligned}$$

so that, after integrating and enforcing the boundary conditions $\kappa_{1t} \to 0$ and $\kappa_{2t} \to 0$, we have

$$\begin{aligned}
\kappa_{1t} &= \int_t^\infty \left(\eta_s^*\Omega_s^2 + \lambda_s^2\nu_s^2\right)ds \\
&= \left(\Omega_t - \Omega_\infty\right) + \left(\Sigma_t - \Sigma_\infty\right) \\
&= \Omega_t + \Sigma_t \\
\kappa_{2t} &= -\mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}c(\eta_s^*)ds\right]
\end{aligned}$$

where the second equality integrates the differential equations for $\Omega$ and $\Sigma = \Omega + \Psi$, and the third equality uses the conjecture (later verified) that $\Omega_t \to 0$ and $\Sigma_t \to 0$ in the limit.

### A.1.3 Equilibrium conditions and derivation

Bringing together the results from the previous two subsections, to conclude the construction of equilibrium we must characterize functions $(\lambda, \beta, \eta; \Omega, \Psi)$ that solve the system

$$\lambda_t = \frac{\beta_t\Psi_t}{\nu_t^2} \tag{55}$$

$$E_t\left[d\left(1/\lambda_t\right)\right] = \frac{r}{\lambda_t}dt \tag{56}$$

$$d\Psi_t = (\eta_t \Omega_t^2 - \nu_t^2 \lambda_t^2) dt \tag{57}$$

$$\eta_t = f\left(\frac{1}{2\lambda_t} \Omega_t^2\right) \tag{58}$$

$$d\Omega_t = -\eta_t \Omega_t^2 dt. \tag{59}$$

As in Collin-Dufresne and Fos (2016), the key step is solving for $(1/\lambda_t, \Psi_t)$. Recall that $\Sigma_t = \Psi_t + \int_t^\infty \eta_s \Omega_s^2 ds$. The equations for $(1/\lambda_t, \Psi_t)$ can be written concisely as

$$\mathbb{E}\left[d\left(\frac{1}{\lambda}\right)_t\right] = r\left(\frac{1}{\lambda}\right)_t dt \tag{60}$$

$$d\Sigma_t = -\lambda_t^2 \nu_t^2 dt. \tag{61}$$

For process $G_t$ to be determined, write $\lambda_t = e^{-rt}\sqrt{\frac{\Sigma_t}{G_t}}$, which decouples the equations as

$$\mathbb{E}[d\sqrt{G_t}] = -\frac{1}{2} e^{-2rt} \frac{\nu_t^2}{\sqrt{G}} dt \tag{62}$$

$$\Leftrightarrow \sqrt{G_t} = \mathbb{E}_t\left[\int_t^\infty \frac{1}{2} e^{-2rs} \frac{\nu_s^2}{\sqrt{G_s}} ds\right]. \tag{63}$$

and

$$\frac{d\Sigma_t}{\Sigma_t} = -e^{-2rt} \frac{\nu_t^2}{G_t} dt, \tag{64}$$

with boundary condition on $G_t$ to be determined. The expression for $\Sigma$ in the Proposition now follows directly from integrating eq. (64).

We have now specified all of $(\lambda, \beta, \eta; \Omega, \Psi)$, up to the boundary condition on $G_t$ as $t \to \infty$. The transversality condition for the trader is

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-rt} J(t, M_t)\right] = \lim_{t\to\infty} \mathbb{E}\left[e^{-rt}\left(\frac{M_t^2 + \kappa_{1t}}{2\lambda_t} + \kappa_{2t}\right)\right].$$

which it is straightforward to show is satisfied under our maintained boundary conditions $G_t \to 0, \kappa_{1t} \to 0, \kappa_{2t} \to 0$.

To conclude the proof, it remains to demonstrate existence of a unique, positive solution $\Omega$ to the initial value problem and show that $\lim_{t\to\infty} \Omega_t = 0$, as conjectured. The following Lemma characterizes $\Omega_t$ and completes the proof:

**Lemma 2.** *There exists a unique, positive stochastic process* $\Omega_t$ *that satisfies* $d\Omega_t = -f\left(\frac{\Omega_t^2}{2\lambda_t}\right)\Omega_t^2 dt$, $\Omega_0 = var(V|\mathcal{F}_t^I) = \Sigma_0 > 0$. *This process satisfies* $\lim_{t\to\infty} \Omega_t = 0$.

*Proof.* Fix any state (i.e., any path of $\nu$) and consider the initial value problem $\frac{d\Omega_t}{dt} = -f\left(\frac{\Omega_t^2}{2\lambda_t}\right)\Omega_t^2, \Omega_0 = \Sigma_0$. Clearly any solution $\Omega_t$, if one exists, is weakly positive since $\Omega_0 > 0$ and at any point at which $\Omega_t = 0$ we have $\frac{d\Omega}{dt} = 0$ so that $\Omega_t$ cannot become strictly negative. Define the function $F(t,x) = -f\left(\frac{1}{2\lambda_t}x^2\right)x^2$ and write the differential equation as $\frac{d\Omega}{dt} = F(t,\Omega)$. Pick any $\varepsilon > 0$. On the open set $D = (-\varepsilon,\infty) \times (-\varepsilon,\infty)$, the function $F$ is locally Lipschitz continuous with respect to its second argument since its derivative with respect to $x$ is continuous. Continuity of the derivative holds since the function $f = (c')^{-1}$ has derivative $f'(x) = 1/c'(c^{-1}(x))$, which is continuous by our assumptions on the cost function $c$. Since our initial condition $(t_0, x_0) = (0, \text{var}(V_0|S_0))$ lies in $D$, it follows from Walter (1998) (Ch. 2, Sect. 6, Thm. VII) that there exists a unique solution to the initial value problem on an interval $[0, b)$ where $0 < b \leq \infty$. To ensure that $b = \infty$ and therefore a solution exists at all times, we must rule out solutions that blow up in finite time. However, since $\frac{d\Omega}{dt} \leq 0$ and $\Omega$ is positive, we know this cannot occur. Hence, there exists a unique, positive solution $\Omega(t)$ to the initial value problem. Since this is true for any state, we conclude that *process* $\Omega_t$ so-defined state by state exists and is unique.

It remains to show that $\lim_{t\to\infty}\Omega_t = 0$. Fix any state. Since $\Omega_t$ is positive and decreasing it has a well defined limit, $m \geq 0$ (which, in general, depends on the state). Let $m = \lim_{t\to\infty}\Omega_t$ and suppose $m > 0$. Then for any $\varepsilon : 0 < \varepsilon < m$ we have $\Omega_t > m - \varepsilon > 0$ for sufficiently large $t$, which further implies that for such $t$,

$$\begin{aligned}\frac{d\Omega_t}{dt} &= -f\left(\frac{1}{2\lambda_t}\Omega_t^2\right)\Omega_t^2 \\ &< -f\left(\frac{1}{2\lambda_t}(m-\varepsilon)^2\right)(m-\varepsilon)^2.\end{aligned}$$

Further, note that

$$\begin{aligned}\frac{1}{\lambda_t} &= e^{rt}\sqrt{\frac{G_t}{\Sigma_t}} \\ &= e^{rt}\sqrt{\frac{G_t}{\Sigma_0 e^{-\int_0^t e^{-2rs}\frac{\nu_s^2}{G_s}ds}}} \\ &\geq e^{rt}\sqrt{\frac{G_t}{\Sigma_0}} \\ &\geq e^{rt}\sqrt{\frac{e^{-2rt}\frac{\underline{\nu}}{2r}}{\Sigma_0}} \\ &= \sqrt{\frac{1}{2r}\frac{\underline{\nu}}{\Sigma_0}} > 0,\end{aligned}$$

so that in fact

$$\begin{aligned}\frac{d\Omega_t}{dt} &< -f\left(\frac{1}{2\lambda_t}(m-\varepsilon)^2\right)(m-\varepsilon)^2 \\ &\leq -f\left(\frac{1}{2}\sqrt{\frac{1}{2r}\frac{\nu}{\Sigma_0}}(m-\varepsilon)^2\right)(m-\varepsilon)^2 \\ &\equiv -C\end{aligned}$$

for constant $C > 0$, which is bounded away from zero. Hence for $t' > t$, with $t$ sufficiently large we have

$$\begin{aligned}\Omega_{t'} &= \Omega_t - \int_t^{t'} f\left(\frac{1}{2\lambda_s}\Omega_s^2\right)\Omega_s^2 ds \\ &\leq \Omega_t - (t'-t)C\end{aligned}$$

which is strictly negative when $t'$ is also sufficiently large. This contradicts the positivity of $\Omega$, which establishes $m = 0$. □

## A.2 Proof of Lemma 1

In the general case in which $\nu$ is uniformly bounded, the existence of the $G_t$ process and the stated bounds follow from the analogous arguments to those in Collin-Dufresne and Fos (2016).

To establish the PDE representation of $G_t$, note that, formally, the backward stochastic differential equation that characterizes $y_t \equiv \sqrt{G_t}$ is

$$dy_t = -\frac{1}{2}e^{-2rt}\frac{\nu_t^2}{y_t}dt - \Lambda_t dW_{\nu t}, \qquad y_t \to 0$$

and a solution to this equation is a pair of processes $(y, \Lambda)$. Applying Ito's Lemma to an arbitrary (sufficiently smooth) function $\gamma(t, \nu_t)$ and matching coefficients establishes the PDE representation, with the boundary condition on $\gamma$ pinned down by the boundary condition on $y_t$.

In the case of deterministic drift, consider the process $G_t$ defined by $\sqrt{G_t} = \sqrt{B(t)}\nu_t$ for deterministic function $B$ to be determined. Plugging $\gamma(t, \nu) = \sqrt{B(t)}\nu$ into the PDE in the

Lemma yields

$$
\begin{aligned}
&\gamma_t + \nu\mu_\nu(t)\gamma_\nu + \frac{1}{2}\nu^2\sigma_\nu^2(t,\nu)\gamma_{\nu\nu} + \frac{1}{2}\frac{e^{-2rt}\nu^2}{\gamma} = 0 \\
\Leftrightarrow &\frac{1}{2}\frac{B'(t)}{\sqrt{B(t)}}\nu + \nu\mu_\nu(t)\sqrt{B(t)} + \frac{1}{2}\frac{e^{-2rt}\nu^2}{\sqrt{B(t)}\nu} = 0 \\
\Leftrightarrow &B'(t) + 2\mu_\nu(t)B(t) + e^{-2rt} = 0
\end{aligned}
$$

The general solution to this ordinary differential equation is

$$
B(t) = e^{\int_0^t -2\mu_\nu(u)du}\left(C - \int_0^t e^{-2rs+\int_0^s 2\mu_\nu(u)du}ds\right)
$$

where $C$ is an arbitrary constant. The boundary condition on $G_t$ requires $B(t) \to 0$, which implies

$$
\begin{aligned}
\lim_{t\to\infty} B(t) = \lim_{t\to\infty}&\left(e^{\int_0^t -2\mu_\nu(u)du}\left(C - \int_0^t e^{-2rs+\int_0^s 2\mu_\nu(u)du}ds\right)\right) = 0 \\
\Leftrightarrow C = &\int_0^\infty e^{-2rs+\int_0^s 2\mu_\nu(u)du}ds.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
B(t) &= e^{\int_0^t -2\mu_\nu(u)du}\int_t^\infty e^{-2rs+\int_0^s 2\mu_\nu(u)du}ds \\
&= \int_t^\infty e^{-2rs+\int_t^s 2\mu_\nu(u)du}ds \\
&= e^{-2rt}\int_t^\infty e^{-2r(s-t)+\int_t^s 2\mu_\nu(u)du}ds \\
&= e^{-2rt}\int_t^\infty e^{\int_t^s 2(\mu_\nu(u)-r)du}ds.
\end{aligned}
$$

# B Proofs of Results from Section 4

In this subsection, we show that any Markovian equilibrium of our model must feature trade timing indifference. This, together with Proposition 3 implies that there cannot be an equilibrium with costly information acquisition in our framework. We begin by generalizing the model in Section 2 along a few dimensions. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is defined an $(n+1)$-dimensional standard Brownian motion $\bar{W} = (W_1, \ldots, W_n, W_Z)$ with filtration $\mathcal{F}_t^W$, independent random variables $S$ and $T$ and independent $m$-dimensional random vector $\nu_0$. Let $\mathcal{F}_t$ denote the augmentation of the filtration $\sigma(\nu_0, \{\bar{W}_s\}_{\{0\le s\le t\}})$. Suppose that the random variable $T$ is exponentially distributed with rate $r$, and that $S \in \mathcal{S} \equiv \text{Support}(S)$ and $\nu_0$ have finite second moments. Finally, let $W = (W_1, \ldots, W_n)$ denote the first $n$ elements of $\bar{W}$.

There is an $m \geq 0$ dimensional vector of publicly-observable signals $\nu_t = (\nu_{1t}, \ldots, \nu_{mt})$ with initial value $\nu_0$ and which follows

$$d\nu = \mu_\nu(t, \nu)\, dt + \Sigma_\nu(t, \nu)\, dW_t,$$

where $\mu_t = (\mu_{1t}, \ldots, \mu_{mt})$ and $\Sigma_{\nu t} = (\Sigma'_{\nu 1t}, \ldots, \Sigma'_{\nu mt})$ denote the vector of drifts and matrix of diffusion coefficients. Suppose that $\mu$ and $\Sigma_\nu$ are such that there exists a unique strong solution to this set of stochastic differential equations (SDEs).[31] Given knowledge of the signal $S$ and the history of $\nu_t$, the conditional expected value $V_t$ of the risky asset's payoff as of time $t$ is

$$V_t = f(t, \nu_t, S)$$

for some function $f$ that, for each $s \in \mathcal{S}$, is continuously differentiable in $t$ and twice continuously differentiable in $\nu$. We assume further that $f$ is such that $V_t$ is a martingale for an agent informed of $S$.[32]

Denote the noise traders's holdings by $Z_t$, where

$$dZ_t = \sigma_Z(t, \nu, Z)\, dW_{Zt}, \tag{65}$$

with $\sigma_Z(\cdot) > 0$ such that there exists a unique strong solution to this SDE. We explicitly allow for the possibility that the volatility depends on the news process, as well as the current cumulative noise trader holdings.

It is important to note that this setup nests a number of existing settings in the literature. For example, Back and Baruch (2004) consider the case in which $S \in \{0, 1\}$ has a binomial distribution, there are no publicly observable signals, and $V_t = f(t, \nu_t, S) = S$. In the special case of Caldentey and Stacchetti (2010) in which time is continuous and there is no ongoing flow of private information, $S \sim \mathcal{N}(0, \Sigma_0)$ and $V_t = f(t, \nu_t, S) = S$.[33] In contrast

[31] See for instance Theorem 5.2.9 in Karatzas and Shreve (1998) who present Lipschitz and growth conditions on the coefficients that are sufficient to deliver this result in a Markovian setting.

[32] A simple set of sufficient conditions is: for each $s \in \mathcal{S}$, $f_t + f_\nu \cdot \mu + \frac{1}{2} tr(f_{\nu\nu} \Sigma\Sigma') = 0$ and $\mathbb{E}[\int_0^\infty f'_\nu \Sigma\Sigma' f_\nu du] < \infty$, which guarantee that $V$ is a (square integrable) martingale. Note also that because all market participants are risk-neutral, it is without loss of generality, economically, that $V_t$ is a martingale, that $V_t$ represents the conditional expected value of the asset rather than the value itself, and that we treat $V_T$ as the terminal value.

[33] Note that because the strategic trader receives only a "lump" of private information, our model is not

to these earlier models, in which the insider is endowed with private information about the asset value, our focus is on allowing her to acquire information at a time of her choosing. Banerjee and Breon-Drish (2018) also allow for dynamic information acquisition (and entry) and consider a special case in which $S \in \{l, h\}$, $\nu$ follows a geometric Brownian motion with zero drift, and $V_t = f(t, \nu_t, S) = \nu \xi_S$, where $\xi$ is a constant that depends on the realization of $S$. Importantly, however, they assume that the entry decision can be detected by the market maker.

Following the literature, we consider Markovian equilibria in which the asset price is a function of the exogenous public signal $\nu_t$, as well as an arbitrary (but finite) number $\ell$ of endogenous state variables $p_t$ that follow a Markovian diffusion and which keep track of the market maker's beliefs about $S$. In particular, we consider pricing rules of the form $P_t = g(t, \nu_t, p_t)$ where $g$ is continuously differentiable in $t$ and twice continuously differentiable in $(\nu, p)$ and is strictly increasing in both $p$ and $\nu$.[34] There are $\ell > 0$ endogenous state variables $p_t$ with dynamics

$$dp = \alpha(t, p, \nu)\, dt + \Gamma(t, p, \nu)\, dW + \mathbf{1} dY, \tag{66}$$

where $\alpha$ is an $\ell$-dimensional function, $\Omega$ is an $\ell \times m$ matrix function, and $\mathbf{1}$ is an $\ell \times 1$ vector of ones such that there exists a unique strong solution to this SDE when $dY = dX + dZ$ and the trading strategy $X_t$ takes an admissible form (to be detailed below). We normalize $p_{0^-} = 0$. Without loss of generality and to simplify later notation, we also normalize the coefficients on $dY$ to be identically equal to one. Importantly, this does *not* imply that we restrict the price impact of a one unit trade to one dollar. Recalling that the function $g(\cdot)$ maps the state variables to the price, the overall dependence of the price on order flow is captured by $dP = \cdots + g_p \cdot \mathbf{1} dY$ where. Hence "Kyle's lambda" is $g_p \cdot \mathbf{1}$. We emphasize that the function $g$ and the coefficients $\alpha$ and $\Omega$ are equilibrium objects that, given an equilibrium trading strategy, are pinned down by the rationality of the pricing rule.

We also require a condition on the set of trading strategies in order to rule out doubling-type strategies (see, e.g., Back (1992)). In particular, a trading strategy is *admissible* if is a semi-martingale adapted to the strategic trader's filtration, and for all pricing rules satisfying

subject to the Caldentey and Stacchetti (2010) critique that the continuous-time equilibrium is not the limit of corresponding discrete time equilibria. However, all of the results below easily extend to time-varying signal $S_t$ when "acquiring information" entails paying $c$ to either perfectly observe $S_t$ at time $\tau$, or to observe it from from time $\tau$ forward. In fact, we show below that our nonexistence results extend to an analogous discrete-time model if the time between trading rounds is sufficiently small.

[34] The monotonicity condition is a normalization that ensures that increases in $p$ and $\nu$ represent "good news."

the conditions specified above, we have

$$\mathbb{E}\left[\int_0^\infty \left(e^{-ru}(V_u - P_{u^-})\right)^2 d\left[Z,Z\right]_u\right] < \infty$$
$$\mathbb{E}\left[\int_0^\infty \left(e^{-ru}X_{u^-}\right)^2 d\left[V,V\right]_u\right] < \infty$$

where $[\cdot]$ denotes the quadratic (co)variation.

We now formalize our assumption on the HJB equation.

**Assumption 1.** *Suppose that in any conjectured equilibrium, for each $s \in support(S)$, there exists a function $J^s(t,\nu,p)$ is continuously differentiable in $t$, twice continuously differentiable in $(\nu,p)$, that satisfies the HJB equation*[35]

$$0 = \sup_\theta \left\{ \begin{array}{c} -rJ^s + J^s_t + J^s_\nu \cdot \mu + J^s_p \cdot (\alpha + \mathbf{1}\theta) \\ +\frac{1}{2}tr\left(J^s_{\nu\nu}\Sigma\Sigma'\right) + \frac{1}{2}tr\left(J^s_{pp}(\Gamma\Gamma' + \mathbf{1}\sigma_z^2\mathbf{1}')\right) \\ +tr\left(J_{\nu p}\Gamma\Sigma'\right) + \theta\left(V_t - P_{t^-}\right) \end{array} \right\}, \tag{67}$$

*and when the order flow is generated by the conjectured equilibrium strategy, the integrability conditions*

$$\mathbb{E}\left[\int_0^\infty e^{-2rs} J_p'(s^-)\Gamma\Gamma' J_p(s^-)\, ds\right] < \infty$$
$$\mathbb{E}\left[\int_0^\infty e^{-2rs} J_\nu'(s^-)\Sigma\Sigma' J_\nu(s^-)\, ds\right] < \infty,$$

*and the transversality condition*

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-rt}J(t,\nu_t,p_t)\right] = 0$$

*hold.*

Note that, if the HJB equation does in fact characterize the value function, the transversality condition implicitly provides information on what kinds of trading strategies are consistent with equilibrium (recalling that trades drive the $p_t$ variables), as it says that an optimal strategy exhausts all profitable trading opportunities if the trading game were to continue indefinitely (i.e., if $T$ tended to infinity). The integrability conditions, on the other hand, are essentially technical. Note also that Assumption 1 applies to the equilibria in existing models in the literature (e.g., Back and Baruch (2004), the continuous-time case

[35] When applied to a vector or matrix, $'$, denotes the transpose.

of Caldentey and Stacchetti (2010), and the fixed-horizon, continuous-time model of Kyle (1985) with the appropriate modification).

We now establish that in any such equilibrium, if it were to exist, an informed trader's optimal trading strategy is absolutely continuous i.e., $dX_t = \theta(\cdot)\,dt$, where $\theta(\cdot)$ denotes the trading rate, and her value function is characterized by the HJB equation. Importantly, we also show that such an equilibrium must feature trade timing indifference. Note that the result below applies to both pure strategy equilibria in which information is immediately acquired and mixed strategy equilibria.

**Proposition 5.** *Suppose Assumption 1 holds. If there exists an overall equilibrium, then any optimal trading strategy for an informed trader is absolutely continuous and the value function for an informed trader is the solution to the HJB equation in Assumption 1, subject to the integrability and transversality conditions. Moreover, such an equilibrium must feature* ***trade timing indifference****.*

The optimality of the smooth trading strategy is established in the Appendix, and extends the arguments in Kyle (1985) and Back (1992) to our setting. Intuitively, if an informed trader does not trade smoothly she reveals her information too quickly. Moreover, the proof in the Appendix establishes that the $J^s$ are all solutions of the HJB equation (67). Because this equation is linear in $\theta$, and $\theta$ is unconstrained, it follows that the sum of the coefficients on $\theta$ must be identically zero and therefore the sum of the remaining terms must also equal zero i.e.,

$$\begin{array}{c} -rJ^s + J^s_t + J^s_\nu \cdot \mu + J^s_p \cdot \alpha \\ +\frac{1}{2}\mathrm{tr}\left(J^s_{\nu\nu}\Sigma\Sigma'\right) + \frac{1}{2}\mathrm{tr}\left(J^s_{pp}(\Gamma\Gamma' + \mathbf{1}\sigma_z^2\mathbf{1}')\right) + \mathrm{tr}\left(J_{\nu p}\Gamma\Sigma'\right) \end{array} = 0 \tag{68}$$

But the above is simply the expected differential of the value function of an informed investor under the assumption that her trading rate at $t$ is zero i.e., $\theta^s_t = 0$. This establishes trade timing indifference.

## B.1 Proof of Proposition 5.

We will make use the following, equivalent, expressions for the trader's terminal wealth, the second of which follows from the integration by parts formula for semi-martingales

$$\begin{aligned} W_T &= (V_T - P_T)X_T + \int_0^T X_{s^-} dP_s \\ &= \int_0^\infty X_{s^-} dV_s + \int_0^T V_{s^-} dX_s + [X, V]_T - \int_0^T P_{s-} dX_s - [X, P]_T \end{aligned}$$

$$= \int_0^T (V_{s^-} - P_{s^-})dX_s + \int_0^T d[X, V-P]_s + \int_0^T X_{s^-}dV_s$$

where, for any $s$, $[\cdot]_s$ denotes the quadratic (co)variation over the interval $[0, s]$. Hence, under any information set that does not include $T$, we have from the independence of $T$

$$\mathbb{E}[W_t|\cdot] = \mathbb{E}\left[\int_0^\infty e^{-rs}(V_{s^-} - P_{s^-})dX_s + \int_0^\infty e^{-rs}d[X, V-P]_s + \int_0^\infty e^{-rs}X_{s^-}dV_s \middle| \cdot\right]$$

The proof now proceeds in a way similar to that of Lemma 2 in Back (1992) and follows it closely. We show that the solution to the HJB equation provides an upper bound on the expected profit from any admissible trading strategy and that any absolutely continuous trading strategy that induces the value function to satisfy the transversality and integrability conditions achieves equality in upper the bound. For the rest of this proof, we suppress all arguments of the function $J$, other than time, where no confusion will result.

Without loss of generality, suppose the trader becomes informed at $t = 0$. The case in which she becomes informed at any other date is analogous. Consider an arbitrary semi-martingale strategy $X_s$. The generalized Ito's formula (Protter (2003), Thm. 33, Ch. 2) for semi-martingales implies that under the smoothness assumptions on $J$ (suppressing the $\nu$ and $p$ arguments for brevity), and recalling that jumps, if any, come from the $p$ process

$$\begin{aligned} e^{-rt}J(t) - J(0^-) = & \int_0^t e^{-rs}(-rJ(s^-) + J_s(s^-))\,ds + \int_0^t e^{-rs}J_\nu(s^-)\cdot d\nu + \int_0^t e^{-rs}J_p(s^-)\cdot dp \\ & + \frac{1}{2}\int_0^t e^{-rs}\text{tr}(J_{\nu\nu}(s^-)d[\nu^c,\nu^c]) + \frac{1}{2}\int_0^t e^{-rs}\text{tr}(J_{pp}(s^-)d[p^c,p^c]) \\ & + \int_0^t e^{-rs}\text{tr}(J_{\nu p}(s^-)d[p^c,\nu^c]) + \sum_{0\le s\le t} e^{-rs}\left(J(s) - J(s^-) - J_p(s^-)\cdot\Delta p_s\right), \end{aligned}$$

where the $c$ superscript denotes the continuous, local martingale portion of a given process.

The quadratic variations of the continuous portions can be written

$$\begin{aligned} [\nu^c,\nu^c]_t &= \int_0^t \Sigma\Sigma' ds \\ [p^c,p^c]_t &= \int_0^t d[\Gamma W,\Gamma W]_s ds + 2\int_0^t d[\Gamma W, \mathbf{1}Y^c]_s ds + \int_0^t d[\mathbf{1}Y^c,\mathbf{1}Y^c]_s ds \\ &= \int_0^t \Gamma\Gamma' ds + 2\int_0^t \Gamma d[W, X^c]_s \mathbf{1}' ds + \int_0^t \mathbf{1}d[Y^c,Y^c]_s\mathbf{1}' ds \\ [p,\nu]_t^c &= \int_0^t d[\Gamma W,\Sigma W]_s ds + \int_0^t d[\mathbf{1}Y^c,\Sigma W]_s ds \end{aligned}$$

$$= \int_0^t \Gamma\Sigma' ds + \int_0^t \mathbf{1} d[X^c, W]_s \Sigma'$$

We have $[Y^c, Y^c]_t = [X^c, X^c]_t + 2[X^c, Z]_t + \int_0^t \sigma_Z^2 ds$.Therefore, the quadratic variations become

$$\begin{aligned}
[\nu^c, \nu^c]_t &= \int_0^t \Sigma\Sigma' ds \\
[p^c, p^c]_t &= \int_0^t \Gamma\Gamma' ds + 2\int_0^t \Gamma d[W, X^c]_s \mathbf{1}' ds + \int_0^t \mathbf{1} d[X^c, X^c]_s \mathbf{1}' ds \\
&\quad + 2\int_0^t \mathbf{1} d[X^c, Z]_s \mathbf{1}' ds + \int_0^t \mathbf{1}\sigma_Z^2 \mathbf{1}' ds \\
[p^c, \nu^c]_t &= \int_0^t \Gamma\Sigma' ds + \int_0^t \mathbf{1} d[X^c, W]_s \Sigma' ds.
\end{aligned}$$

Returning to the expression for $J(t) - J(0)$ above and also using the fact $\Delta p_s = \mathbf{1}\Delta X$ gives

$$\begin{aligned}
e^{-rt}J(t) - J(0^-) = &\int_0^t e^{-rs} \Big( -rJ(s^-) + J_s(s^-) + J_\nu(s^-) \cdot \mu + J_p(s^-) \cdot \alpha + \frac{1}{2}\mathrm{tr}(J_{\nu\nu}\Sigma\Sigma') \\
&\quad + \mathrm{tr}(J_{\nu p}\Gamma\Sigma') + \frac{1}{2}\mathrm{tr}(J_{pp}(\Gamma\Gamma' + \mathbf{1}\sigma_Z^2\mathbf{1}')) \Big) \, ds \\
&+ \int_0^t e^{-rs} J_p(s^-) \cdot \mathbf{1} dX + \int_0^t e^{-rs} J_p(s^-) \left(\Gamma dW + \mathbf{1} dZ\right) + \int_0^t e^{-rs} J_\nu(s^-) \Sigma dW \\
&+ \frac{1}{2}\int_0^t e^{-rs} \mathrm{tr}(J_{pp}(s^-)(\mathbf{1}d[X^c X^c]\mathbf{1}' + 2\Gamma d[W, X^c]\mathbf{1}' + 2\mathbf{1}d[X^c, Z]\mathbf{1}')) \\
&+ \int_0^t e^{-rs} \mathrm{tr}(J_{\nu p}(s^-)\mathbf{1}d[X^c, W]\Sigma') + \sum_{0 \le s \le t} e^{-rs} \left( J(s) - J(s^-) - J_p(s^-) \cdot \mathbf{1}\Delta X_s \right).
\end{aligned}$$

The HJB equation implies

$$J_p \cdot \mathbf{1} + V_t - P_{t^-} = 0 \tag{69}$$

$$\begin{aligned}
&- rJ(s^-) + J_t(s^-) + J_\nu(s^-) \cdot \mu + J_p(s^-) \cdot \alpha + \frac{1}{2}\mathrm{tr}(J_{\nu\nu}\Sigma\Sigma') \\
&\quad + \mathrm{tr}(J_{\nu p}\Gamma\Sigma') + \frac{1}{2}\mathrm{tr}(J_{pp}(\Gamma\Gamma' + \mathbf{1}\sigma_Z^2\mathbf{1}')) = 0
\end{aligned} \tag{70}$$

Substituting first eq. (70) into the previous expression gives

$$e^{-rt}J(t) - J(0^-) = \int_0^t e^{-rs} J_p(s^-) \cdot \mathbf{1} dX + \int_0^t e^{-rs} J_p(s^-) \left(\Gamma dW + \mathbf{1} dZ\right) + \int_0^t e^{-rs} J_\nu(s^-) \Sigma dW$$

$$+\frac{1}{2}\int_0^t e^{-rs}\mathrm{tr}(J_{pp}(s^-)(\mathbf{1}d[X^cX^c]\mathbf{1}' + 2\Gamma d[W,X^c]\mathbf{1}' + 2\mathbf{1}d[X^c,Z]\mathbf{1}'))$$
$$+\int_0^t e^{-rs}\mathrm{tr}(J_{\nu p}(s^-)\mathbf{1}d[X^c,W]\Sigma') + \sum_{0\leq s\leq t} e^{-rs}\left(J(s) - J(s^-) - J_p(s^-)\cdot\mathbf{1}\Delta X_s\right).$$

Now substituting eq. (69)

$$e^{-rt}J(t) - J(0^-) = -\int_0^t e^{-rs}(V_s - P_{s^-})(dX + dZ) + \int_0^t e^{-rs}J_p(s^-)\Gamma dW + \int_0^t e^{-rs}J_\nu(s^-)\Sigma dW$$
$$+\frac{1}{2}\int_0^t e^{-rs}\mathrm{tr}(J_{pp}(s^-)(\mathbf{1}d[X^cX^c]\mathbf{1}' + 2\Gamma d[W,X^c]\mathbf{1}' + 2\mathbf{1}d[X^c,Z]\mathbf{1}'))$$
$$+\int_0^t e^{-rs}\mathrm{tr}(J_{\nu p}(s^-)\mathbf{1}d[X^c,W]\Sigma') + \sum_{0< s\leq t} e^{-rs}\left(J(s) - J(s^-) - (P_{s^-} - V_s)\Delta X_s\right).$$

Rearrange

$$\int_0^t e^{-rs}(V_s - P_{s^-})dX - J(0^-)$$
$$= -e^{-rt}J(t) - \int_0^t e^{-rs}(V_s - P_{s^-})dZ + \int_0^t e^{-rs}J_p(s^-)\Gamma dW + \int_0^t e^{-rs}J_\nu(s^-)\Sigma dW$$
$$+\frac{1}{2}\int_0^t e^{-rs}\mathrm{tr}(J_{pp}(s^-)(\mathbf{1}d[X^cX^c]\mathbf{1}' + 2\Gamma d[W,X^c]\mathbf{1}' + 2\mathbf{1}d[X^c,Z]\mathbf{1}'))$$
$$+\int_0^t e^{-rs}\mathrm{tr}(J_{\nu p}(s^-)\mathbf{1}d[X^c,W]\Sigma') + \sum_{0\leq s\leq t} e^{-rs}\left(J(s) - J(s^-) - (P_{s^-} - V_s)\Delta X_s\right)$$

Use the cyclic property of the trace and the fact that the trace of a scalar is itself to simplify the traces to matrix multiplication

$$\int_0^t e^{-rs}(V_s - P_{s^-})dX - J(0^-)$$
$$= -e^{-rt}J(t) - \int_0^t e^{-rs}(V_s - P_{s^-})dZ + \int_0^t e^{-rs}J_p(s^-)\Gamma dW + \int_0^t J_\nu(s^-)\Sigma dW$$
$$+\int_0^t e^{-rs}\frac{1}{2}d[X^c,\mathbf{1}X^c]J_{pp}(s^-)\mathbf{1} + d[X^c,\Gamma W]J_{pp}(s^-)\mathbf{1} + d[X^c,\mathbf{1}Z]J_{pp}(s^-)\mathbf{1}))$$
$$+\int_0^t e^{-rs}d[X^c,\Sigma W]J_{\nu p}(s^-)\mathbf{1} + \sum_{0\leq s\leq t} e^{-rs}\left(J(s) - J(s^-) - (P_{s^-} - V_s)\Delta X_s\right)$$

Now, add $\int_{[0,t)} e^{-rs} d[X, V-P]_s + \int_{[0,t]} e^{-rs} X_{s^-} dV_s$ to both sides

$$
\begin{aligned}
&\int_0^t e^{-rs}(V_s - P_{s^-})dX + \int_0^t e^{-rs} d[X, V-P]_s + \int_{[0,t]} e^{-rs} X_{s^-} dV_s - J(0) \\
&= -e^{-rt}J(t) - \int_0^t e^{-rs}(V_s - P_{s^-})dZ + \int_0^t e^{-rs} J_p(s^-)\Gamma dW + \int_0^t e^{-rs} J_\nu(s^-)\Sigma dW \\
&+ \int_0^t e^{-rs}(\frac{1}{2} d[X^c, \mathbf{1}X^c] J_{pp}(s^-)\mathbf{1} + d[X^c, \Gamma W] J_{pp}(s^-)\mathbf{1} + d[X^c, \mathbf{1}Z] J_{pp}(s^-)\mathbf{1}) \\
&+ \int_0^t e^{-rs} d[X^c, \Sigma W] J_{\nu p}(s^-)\mathbf{1} + \sum_{0 \le s \le t} e^{-rs}\left(J(s) - J(s^-) - (P_{s^-} - V_s)\Delta X_s\right) \\
&\quad + \int_0^t e^{-rs} d[X, V-P]_t + \int_0^t e^{-rs} X_{s^-} dV_s
\end{aligned}
\tag{71}
$$

Note that the continuous, local martingale portion of $V_t - P_t$ (as differentials) is

$$
(f_\nu - g_\nu) \cdot \Sigma dW - g_p \cdot (\Gamma dW + \mathbf{1} dY^c).
$$

Hence

$$
[X^c, (V-P)^c]_t = \int d[X, (f_\nu - g_\nu) \cdot \Sigma W - g_p \cdot \Gamma W]^c - \int d[X, g_p \cdot \mathbf{1}X]^c - \int d[X, g_p \cdot \mathbf{1}Z]^c
$$

From eq. (69) we have

$$
\begin{aligned}
f_\nu - g_\nu &= -J'_{p\nu}\mathbf{1} = -J_{\nu p}\mathbf{1} \\
g_p &= J_{pp}\mathbf{1} > 0,
\end{aligned}
$$

where the inequality follows from the assumption that $g$ is increasing in the endogenous state variables.

Plugging in to the expression for the quadratic variation

$$
\begin{aligned}
[X^c, (V-P)^c]_t &= \int d[X, -\mathbf{1}' J_{p\nu}\Sigma W - \mathbf{1}' J_{pp}\Gamma W]^c \\
&\quad - \int d[X, \mathbf{1}' J_{pp}\mathbf{1}X]^c - \int d[X, \mathbf{1}' J_{pp}\mathbf{1}Z]^c \\
&= \int -d[X, W]^c \Sigma' J_{\nu p}\mathbf{1} - \int d[X, W]^c \Gamma' J_{pp}\mathbf{1} \\
&\quad - \int d[X, X]^c \mathbf{1}' J_{pp}\mathbf{1} - \int d[X, Z]^c \mathbf{1}' J_{pp}\mathbf{1}
\end{aligned}
\tag{72}
$$

Plugging eq. (72) back into eq. (71) and again using $\Delta(V_s - p_s) = -\Delta p_s = -\mathbf{1}\Delta X$ gives

$$\begin{aligned}
&\int_0^t e^{-rs}(V_s - P_{s^-})dX + \int_0^t e^{-rs}d[X, V-P]_s + \int_0^t e^{-rs}X_{s^-}dV_s - J(0) \\
&= -e^{-rt}J(t) - \int_0^t e^{-rs}(V_s - P_{s^-})dZ + \int_0^t e^{-rs}J_p(s^-)\Gamma dW + \int_0^t e^{-rs}J_\nu(s^-)\Sigma dW + \int_0^t e^{-rs}X_{s^-}dV_s \\
&\quad - \frac{1}{2}\int_0^t e^{-rs}d[X^c, X^c]\mathbf{1}'J_{pp}\mathbf{1} + \sum_{0<s\leq t} e^{-rs}\left(J(s) - J(s^-) - (P_s - V_s)\Delta X_s\right)
\end{aligned}$$

Now, using (69) in the jump term and rearranging to group the stochastic integrals

$$\begin{aligned}
&\int_0^t e^{-rs}(V_s - P_{s^-})dX + \int_0^t e^{-rs}[X, V-P]_s + \int_0^t e^{-rs}X_{s^-}dV_s - J(0^-) \\
&= -e^{-rt}J(t) - \frac{1}{2}\int_0^t e^{-rs}d[X^c, X^c]\mathbf{1}'J_{pp}\mathbf{1} + \sum_{0\leq s\leq t} e^{-rs}\left(J(s) - J(s^-) - J_p\cdot\mathbf{1}\Delta X_s\right) \\
&\quad - \int_0^t e^{-rs}(V_s - P_{s^-})dZ + \int_0^t e^{-rs}J_p(s^-)\Gamma dW + \int_0^t e^{-rs}J_\nu(s^-)\Sigma dW + \int_0^t e^{-rs}X_{s^-}dV_s
\end{aligned} \tag{73}$$

Let $x(t) = -\int_0^t e^{-rs}(V_s - P_{s^-})dZ + \int_0^t e^{-rs}J_p(s^-)\Gamma dW + \int_0^t e^{-rs}J_\nu(s^-)\Sigma dW + \int_{[0,t]} e^{-rs}X_{s^-}dV_s$ denote the stochastic integrals. Recall that by assumption, $V_s$ is a martingale. Owing to the the admissibility condition on trading strategies and integrability conditions on $J$, $x(t)$ is a square integrable martingale. Hence, we can collect the stochastic integrals in the previous expression into a single martingale term, $x(t)$

$$\begin{aligned}
&\int_0^t e^{-rs}(V_s - P_{s^-})dX_s + \int_{[0,t]} e^{-rs}d[X, V-P]_s + \int_0^t e^{-rs}X_{s^-}dV_s - J(0^-) \\
&= -e^{-rt}J(t) - \frac{1}{2}\int_0^t e^{-rs}d[X^c, X^c]\mathbf{1}'J_{pp}\cdot\mathbf{1} \\
&\quad + \sum_{0\leq s\leq t} e^{-rs}\left(J(s) - J(s^-) - J_p\cdot\mathbf{1}\Delta X_s\right) + x(t).
\end{aligned} \tag{74}$$

Taking limits, we now have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-rs}(V_s - P_{s^-})dX_s + \int_{[0,\infty)} e^{-rs}d[X, V-P]_s + \int_{[0,\infty)} e^{-rs}X_{s^-}dV_s - J(0^-)\right] \\
&= \mathbb{E}\left[\lim_{t\to\infty}\int_0^t e^{-rs}(V_s - P_{s^-})dX_s + \int_0^t e^{-rs}d[X, V-P]_s + \int_{[0,t]} e^{-rs}X_{s^-}dV_s - J(0^-)\right]
\end{aligned}$$

$$= \mathbb{E}\left[\lim_{t\to\infty} -e^{-rt}J(t) - \tfrac{1}{2}\int_0^t e^{-rs}d[X^c, X^c]\mathbf{1}'J_{pp}\mathbf{1} \right.$$
$$\left. + \sum_{0\le s\le t} e^{-rs}\left(J(s) - J(s^-) - J_p \cdot \mathbf{1}\Delta X_s\right)\right]$$
$$= \lim_{t\to\infty} \mathbb{E}\left[-e^{-rt}J(t) - \tfrac{1}{2}\int_0^t e^{-rs}d[X^c, X^c]\mathbf{1}'J_{pp}\mathbf{1} + \textstyle\sum_{0\le s\le t} e^{-rs}\left(J(s) - J(s^-) - J_p \cdot \mathbf{1}\Delta X_s\right) + x(t)\right]$$
$$= \lim_{t\to\infty} \mathbb{E}\left[-\tfrac{1}{2}\int_0^t e^{-rs}d[X^c, X^c]\mathbf{1}'J_{pp} \cdot \mathbf{1} + \textstyle\sum_{0\le s\le t} e^{-rs}\left(J(s) - J(s^-) - J_p \cdot \mathbf{1}\Delta X_s\right)\right]$$
$$\le 0$$

where the third line uses 74, the fourth line uses (i) the transversality condition on $J$ and $J \ge 0$, (ii) the fact that $x(t)$ is a square integrable martingale and (iii) the monotone convergence theorem, and the next-to-last line uses the transversality condition and the fact that $x(t)$ is a martingale. The final line uses the fact that $J_{pp} \cdot \mathbf{1} = g_p > 0$ by assumption on the pricing rule and that $[X^c, X^c]$ is a positive measure. Note that the inequality holds with equality if and only if the trading strategy is absolutely continuous, in which case $[X^c, X^c] \equiv 0$, $\Delta X \equiv 0$, and $J(s) - J(s^-) \equiv 0$.

This analysis establishes that the expected profit for any trading strategy is no greater than $J(0^-)$. Furthermore, any absolutely continuous strategy that induces the value function to satisfy the integrability and transversality conditions provides expected profit equal to $J(0^-)$ and and is therefore an optimal strategy. Any strategy that is not absolutely continuous and does not lead to a value function satisfying the transversality condition is strictly suboptimal. □

# C Proofs of Results from Section 4.2

## C.1 Non-existence in Discrete Time

Recall the special case of Caldentey and Stacchetti (2010) in which there is a "lump" of initial private information and no ongoing flow of private information. Time is discrete and trade takes place at dates $t_n = n\Delta$ for $n \ge 0$ and $\Delta > 0$. The risky asset pays off $V \sim \mathcal{N}(0, \Sigma_0)$ immediately after trading round $T$, where $T$ is random. Specifically, $T = \eta\Delta$, where $\eta$ is geometrically distributed with failure probability $\rho = e^{-r\Delta}$.[36] The risk-neutral strategic trader observes $V$. Let $x_n$ denote her trade at date $t_n$. There are noise traders who submit iid trades $z_t \sim \mathcal{N}(0, \Sigma_z)$ with $\Sigma_z = \sigma_z^2\Delta$. Let $y_n = x_n + z_n$ denote the time $n$ order

[36]There are at least two different distributions that are often referred to as "the" geometric distribution. The one we use here is supported on the nonnegative integers $n \in \{0, 1, 2, \dots\}$ and has probability mass function $f_n = \rho^n(1-\rho)$ and cdf $F_n = 1 - \rho^{n+1}$.

flow. Competitive risk-neutral market makers set the price $p_n$ in each trading round equal to the conditional expected value. Following Caldentey and Stacchetti (2010), we focus on linear, Markovian equilibria in which the time $t_n$ price depends only on $p_{n-1}$ and $y_n$. Let $\bar{V}_n$ and $\Sigma_n$ denote the market maker's conditional expectation and variance, immediately *before* the time $t_n$ trading round. So, $\bar{V}_n = p_{n-1}$. Finally, set $p_{-1} = \mathbb{E}[V] = 0$.

Caldentey and Stacchetti (2010) show that if the trader is informed of the asset payoff there exists an equilibrium in which the asset price and trading strategy are given by

$$\begin{aligned} p_n(\bar{V}_n, y_n) &= \bar{V}_n + \lambda_n y_n \\ x_n(V, \bar{V}_n) &= \beta_n(V - \bar{V}_n), \end{aligned}$$

the trader's expected profit is

$$\Pi_n(p_{n-1}, V) = \alpha_n(V - p_{n-1})^2 + \gamma_n,$$

and the coefficients are characterized by the difference equations described in the proof of the following result. The ex-ante expected profit from acquiring information immediately before the $t = 0$ trading round is

$$\begin{aligned} \bar{\Pi}_0 \equiv \mathbb{E}[\Pi_0(p_{-1}, V)] - c &= \alpha_0 \mathbb{E}[(V - p_{-1})^2] + \gamma_0 - c \\ &= \alpha_0 \Sigma_0 + \gamma_0 - c. \end{aligned}$$

We would like to compare this to the expected profit if the trader deviates by remaining uninformed for the $n = 0$ trading round, not trading, and then acquiring immediately before round $n = 1$. Supposing that she does so, trades $x = 0$ units at time zero, and then follows the prescribed equilibrium trading strategy in the following rounds, the expected deviation profit is

$$\begin{aligned} \bar{\Pi}_{d0} &= \mathbb{E}[x(V - p_0(\bar{V}_0, x + z_0))] + \mathbb{E}\left[\sum_{n=1}^{\infty} \rho^n (V - p_n) x_n - \rho c\right] && (75) \\ &= \rho\left(\alpha_1(\Sigma_0 + \lambda_0^2 \Sigma_z) + \gamma_1 - c\right). && (76) \end{aligned}$$

The following result establishes that such a deviation is profitable when the time between trading dates $\Delta$ is sufficiently small.

**Proposition 6.** *For all $\Delta > 0$ sufficiently small,*

$$\frac{\bar{\Pi}_{d0} - \bar{\Pi}_0}{\Delta} = \frac{-\frac{1}{2}\frac{(\beta_0^\Delta)^3\Sigma_0^2}{\sigma_z^2\Delta} + (1 - e^{-r\Delta})c}{\Delta} > 0,$$

*and therefore there does not exist an equilibrium in which the trader follows a pure acquisition strategy.*

### C.1.1 Proof of Proposition 6.

Caldentey and Stacchetti (2010) show that there exists an equilibrium in which the asset price and trading strategy are given by

$$\begin{aligned} p_n(\bar{V}_n, y_n) &= \bar{V}_n + \lambda_n y_n \\ x_n(V, \bar{V}_n) &= \beta_n(V - \bar{V}_n), \end{aligned}$$

the trader's expected profit is

$$J_n(p_{n-1}, V) = \alpha_n(V - p_{n-1})^2 + \gamma_n,$$

and the constants are characterized by the difference equations

$$\begin{aligned} \Sigma_{n+1} &= \frac{\Sigma_n\Sigma_z}{\beta_n^2\Sigma_n + \Sigma_z} \\ \beta_{n+1}\Sigma_{n+1} &= \rho\beta_n\Sigma_n\left(\frac{\Sigma_z^2}{\Sigma_z^2 - \beta_n^4\Sigma_n^2}\right) \\ \lambda_n &= \frac{\beta_n\Sigma_n}{\beta_n^2\Sigma_n + \Sigma_z} \\ \alpha_n &= \frac{1 - \lambda_n\beta_n}{2\lambda_n} \\ \rho\gamma_{n+1} &= \gamma_n - \frac{1 - 2\lambda_n\beta_n}{2\lambda_n(1 - \lambda_n\beta_n)}\lambda_n^2\Sigma_z \\ \gamma_0 &= \sum_{k=0}^{\infty}\rho^k\left(\frac{1 - 2\lambda_k\beta_k}{2\lambda_k(1 - \lambda_k\beta_k)}\right)\lambda_k^2\Sigma_z \end{aligned}$$

The ex-ante expected profit from acquiring information immediately before the $t = 0$ trading round is

$$\begin{aligned} \bar{J}_0 \equiv \mathbb{E}[J_0(p_{-1}, V)] - c &= \alpha_0\mathbb{E}[(V - p_{-1})^2] + \gamma_0 - c \\ &= \alpha_0\Sigma_0 + \gamma_0 - c. \end{aligned}$$

We would like to compare this to the expected profit if the trader deviates by remaining uninformed for the $n = 0$ trading round and then acquiring immediately before round $n = 1$. Supposing that she does so, trades $x$ units at time zero, and then follows the prescribed equilibrium trading strategy in the following rounds, the expected profit is

$$\begin{aligned}
&\mathbb{E}[x(V - p_0(\bar{V}_0, x + z_0))] + \mathbb{E}\left[\sum_{n=1}^{\infty} \rho^n (V - p_n) x_n - \rho c\right] \\
&= \mathbb{E}[x(V - \lambda_0(x + z_0))] + \rho\mathbb{E}\left[\sum_{n=1}^{\infty} \rho^n (V - p_n) x_n - c\right] \\
&= \mathbb{E}[x(V - \lambda_0(x + z_0))] + \rho\mathbb{E}[J_1(p_0, V) - c] \\
&= \mathbb{E}[x(V - \lambda_0(x + z_0))] + \rho\mathbb{E}[\alpha_1(V - p_0)^2 + \gamma_1 - c] \\
&= \mathbb{E}[x(V - \lambda_0(x + z_0))] + \rho\mathbb{E}[\alpha_1(V - \lambda_0(x + z_0))^2 + \gamma_1 - c] \\
&= -x^2\lambda_0 + \rho\left(\alpha_1(\Sigma_0 + \lambda_0^2\Sigma_z + \lambda_0^2 x^2) + \gamma_1 - c\right),
\end{aligned}$$

Take $x = 0$. This yields ex-ante deviation profits

$$\bar{J}_{d0} = \rho\left(\alpha_1(\Sigma_0 + \lambda_0^2\Sigma_z) + \gamma_1 - c\right).$$

This deviation is profitable if and only if

$$\begin{aligned}
&\bar{J}_{d0} - \bar{J}_0 > 0 \\
&\iff \rho\left(\alpha_1(\Sigma_0 + \lambda_0^2\Sigma_z) + \gamma_1 - c\right) - (\alpha_0\Sigma_0 + \gamma_0 - c) > 0 \\
&\iff (\rho\alpha_1 - \alpha_0)\Sigma_0 + \rho\gamma_1 - \gamma_0 + \rho\alpha_1\lambda_0^2\Sigma_z + (1 - \rho)c > 0.
\end{aligned}$$

We have

$$\begin{aligned}
\rho\alpha_1 - \alpha_0 &= \rho\frac{1 - \lambda_1\beta_1}{2\lambda_1} - \frac{1 - \lambda_0\beta_0}{2\lambda_0} \\
&= \rho\frac{1 - \frac{\beta_1^2\Sigma_1}{\beta_1^2\Sigma_1 + \Sigma_z}}{2\frac{\beta_1\Sigma_1}{\beta_1^2\Sigma_1 + \Sigma_z}} - \frac{1 - \frac{\beta_0^2\Sigma_0}{\beta_0^2\Sigma_0 + \Sigma_z}}{2\frac{\beta_0\Sigma_0}{\beta_0^2\Sigma_0 + \Sigma_z}} \\
&= \frac{\rho}{2}\frac{\Sigma_z}{\beta_1\Sigma_1} - \frac{1}{2}\frac{\Sigma_z}{\beta_0\Sigma_0} \\
&= \frac{1}{2}\Sigma_z\frac{1}{\beta_0\Sigma_0}\left(\frac{\rho\beta_0\Sigma_0}{\beta_1\Sigma_1} - 1\right) \\
&= \frac{1}{2}\Sigma_z\frac{1}{\beta_0\Sigma_0}\left(\frac{\Sigma_z^2 - \beta_0^4\Sigma_0^2}{\Sigma_z^2} - 1\right)
\end{aligned}$$

$$= -\frac{1}{2}\frac{\beta_0^3\Sigma_0}{\Sigma_z},$$

where the first equality substitutes in from the difference equation for $\alpha_n$, the second equality substitutes from the equation for $\lambda_n$, the third and fourth simplify and collect terms, the fifth equality uses the difference equation for $\beta_{n+1}\Sigma_{n+1}$, and the final equality simplifies and collects terms.

Similarly,

$$\begin{aligned}
\rho\gamma_1 - \gamma_0 &= -\frac{1-2\lambda_0\beta_0}{2\lambda_0(1-\lambda_0\beta_0)}\lambda_0^2\Sigma_z \\
&= -\frac{1}{2}\frac{1-2\lambda_0\beta_0}{1-\lambda_0\beta_0}\lambda_0\Sigma_z \\
&= -\frac{1}{2}\frac{1-2\frac{\beta_0^2\Sigma_0}{\beta_0^2\Sigma_0+\Sigma_z}}{1-\frac{\beta_0^2\Sigma_0}{\beta_0^2\Sigma_0+\Sigma_z}}\lambda_0\Sigma_z \\
&= -\frac{1}{2}\frac{\Sigma_z-\beta_0^2\Sigma_0}{\Sigma_z}\lambda_0\Sigma_z \\
&= -\frac{1}{2}(\Sigma_z-\beta_0^2\Sigma_0)\lambda_0,
\end{aligned}$$

where the first equality uses the difference equation for $\gamma_n$, the second equality cancels a $\lambda_0$, the third equality substitutes for $\lambda_0$, and the last two equalities simplify.

Furthermore, recalling from the calculations for $\rho\alpha_1 - \alpha_0$ that $\rho\alpha_1 = \frac{\rho}{2}\frac{\Sigma_z}{\beta_1\Sigma_1}$ we have

$$\begin{aligned}
\rho\alpha_1\lambda_0^2\Sigma_z &= \frac{\rho}{2}\frac{\Sigma_z}{\beta_1\Sigma_1}\lambda_0^2\Sigma_z \\
&= \frac{1}{2}\frac{\Sigma_z^2}{\beta_0\Sigma_0\left(\frac{\Sigma_z^2}{\Sigma_z^2-\beta_0^4\Sigma_0^2}\right)}\lambda_0^2 \\
&= \frac{1}{2}\frac{1}{\beta_0\Sigma_0}(\Sigma_z^2-\beta_0^4\Sigma_0^2)\lambda_0^2 \\
&= \frac{1}{2}\frac{1}{\beta_0\Sigma_0}(\Sigma_z^2-\beta_0^4\Sigma_0^2)\frac{\beta_0\Sigma_0}{\beta_0^2\Sigma_0+\Sigma_z}\lambda_0 \\
&= \frac{1}{2}(\Sigma_z^2-\beta_0^4\Sigma_0^2)\frac{1}{\beta_0^2\Sigma_0+\Sigma_z}\lambda_0,
\end{aligned}$$

where the second equality substitutes from the difference equation for $\beta_n\Sigma_n$, the third equality simplifies, the fourth equality substitutes for $\lambda_0$, and the final equality simplifies.

Combining the most recent two displayed expressions, we have

$$\begin{aligned}\rho\gamma_1 - \gamma_0 + \rho\alpha_1\lambda_0^2\Sigma_z &= \frac{1}{2}(\Sigma_z^2 - \beta_0^4\Sigma_0^2)\frac{1}{\beta_0^2\Sigma_0 + \Sigma_z}\lambda_0 - \frac{1}{2}(\Sigma_z - \beta_0^2\Sigma_0)\lambda_0 \\ &= \frac{1}{2}\lambda_0\frac{1}{\beta_0^2\Sigma_0 + \Sigma_z}(\Sigma_z^2 - \beta_0^4\Sigma_0^2 - (\Sigma_z - \beta_0^2\Sigma_0)(\beta_0^2\Sigma_0 + \Sigma_z)) \\ &= \frac{1}{2}\lambda_0\frac{1}{\beta_0^2\Sigma_0 + \Sigma_z}(\Sigma_z^2 - \beta_0^4\Sigma_0^2 - \beta_0^2\Sigma_z\Sigma_0 - \Sigma_z^2 + \beta_0^4\Sigma_0^2 + \beta_0^2\Sigma_z\Sigma_0) \\ &= 0.\end{aligned}$$

It follows that

$$\begin{aligned}\bar{J}_{d0} - \bar{J}_0 &= (\rho\alpha_1 - \alpha_0)\Sigma_0 + \rho\gamma_1 - \gamma_0 + \rho\alpha_1\lambda_0^2\Sigma_z + (1-\rho)c \\ &= -\frac{1}{2}\frac{\beta_0^3\Sigma_0^2}{\Sigma_z} + (1-\rho)c. \end{aligned} \tag{77}$$

We would like to study the behavior of the above expression as $\Delta \to 0$.

**Lemma 3.** *There exists a strictly increasing function $\psi$ such that*

$$\beta_0 = \frac{\sqrt{\Sigma_Z}}{\Sigma_0}\psi(\Sigma_0).$$

*Furthermore, $\psi(0) = 0$ and we have*

$$\psi(\Sigma_0) \leq [1-\rho]^{1/4}\sqrt{\Sigma_0}.$$

*Proof.* This proof leans heavily on the Appendix of Caldentey and Stacchetti (2010) but specialized to the case in which there is no flow of private information. As such, we point out only the essential differences in the analysis.

Define

$$A_n = \Sigma_n, \quad B_n = \frac{\beta_n\Sigma_n}{\sqrt{\Sigma_z}}$$

Then the difference equations for $\Sigma_n$ and $\beta_n\Sigma_n$ imply that $(A_{n+1}, B_{n+1}) = F(A_n, B_n)$, where

$$F_A(A_n, B_n) = \frac{A_n^2}{A_n + B_n^2}, \quad F_B(A_n, B_n) = \rho\left[\frac{A_n^2 B_n}{A_n^2 - B_n^4}\right].$$

Note that these are similar to those in Caldentey and Stacchetti (2010), with the exception that there is no +1 term in $F_A$ owing to the absence of a flow of private information. Further, define

$$G_1(A) = 0, \quad G_2(A) = \sqrt{A}[1-\rho]^{1/4}, \quad G_3(A) = \sqrt{A},$$

where the function $G_1$ is defined so that $F_A(A, G_1(A)) = A$ and $G_2(A)$ is such that $F_B(A, G_2(A)) = G_2(A)$. (Note that there is a typo in this definition in Caldentey and Stacchetti (2010) which states $F_B(A, G_2(A)) = B$. However, this cannot hold in general since $B$ is on only one side of the equation.) Finally, $G_3(A)$ is defined so that a point $(A, B)$ is feasible (i.e., leads to a strictly positive value of $\Sigma_{n+1}\beta_{n+1}$ in its difference equation) if and only if $B < G_3(A)$.

These curves divide $\mathbb{R}^2_+$ into three mutually exclusive regions, the union of which comprises all of $\mathbb{R}^2_+$. First, define the infeasible region $R_5 = \{(A, B) : A \geq 0, B \geq G_3(A)\}$. Second, define region $R_1 = \{(A, B) : A \geq 0, G_2(A) < B < G_3(A)\}$. In this region, $F(A, B)$ is always to the left and higher than $(A, B)$ and the given expression for $F$ implies that starting the iteration $(A_{n+1}, B_{n+1}) = F(A_n, B_n)$ in $R_1$ will eventually lead the sequence to enter the infeasible region $R_5$. Finally, define $R_2 = \{(A, B) : A \geq 0, 0 = G_1(A) < B \leq G_2(A)\}$. Note that any sequence $(A_n, B_n)$ always remains feasible, as shown by Caldentey and Stacchetti (2010). Hence, any candidate $(A_0, B_0)$ must lie in $R_2$.

To put the above more clearly in the setting of Figure 1 in Caldentey and Stacchetti (2010), note that in our case, the function $G_1(A)$ is shifted identically downward to zero. This completely eliminates the regions $R_3$ and $R_4$ in their plot. Furthermore, the stationary point $(\hat{A}, \hat{B})$ in our case is defined by the point at which $G_1(A)$ and $G_2(A)$ intersect. This point is precisely $(0, 0)$. That is, with no flow of information to the insider, in the stationary limit the trader perfectly reveals her information and the market maker faces no residual uncertainty.

To complete the proof, we need to find a curve $\mathcal{C} \subset R_2$ such that $(0, 0) \in \mathcal{C}$ and $F(\mathcal{C}) \subset \mathcal{C}$. Note further that because for sequences in $R_2$, we have $(A_{n+1}, B_{n+1}) = (F_A(A_n, B_n), F_B(A_n, B_n)) < (A_n, B_n)$ we know that such a curve must be strictly increasing. Furthermore, because $F$ is continuous we know that such a curve exists. This curve can be defined by an increasing function $0 \leq \psi(A) \leq G_2(A)$ with $\psi(0) = 0$ so that $\mathcal{C} = \{(A, B) : A \geq 0, B = \psi(A)\}$.

Clearly if we take $B_0 = \psi(A_0)$ then the associated sequence always lies in $\mathcal{C}$ and we have $(A_n, B_n) \downarrow 0$, the stationary point. Hence, returning to the definitions of $A_0$ and $B_0$, this implies that we need to set

$$\begin{aligned} &\frac{\beta_0 \Sigma_0}{\sqrt{\Sigma_Z}} = \psi(\Sigma_0) \\ &\Rightarrow \beta_0 = \Psi(\Sigma_0) \equiv \frac{\sqrt{\Sigma_Z}}{\Sigma_0}\psi(\Sigma_0). \end{aligned}$$

The claimed inequality holds because it was shown above that $0 \leq \psi(A) \leq G_2(A)$. $\square$

We will now proceed with analyzing the behavior of the deviation profit in eq. (77) as $\Delta$ shrinks. To make clear the dependence of the various coefficients on $\Delta$, we write, e.g., $\beta_0^\Delta$ as applicable in the following. Recall that for a positive function of $\Delta > 0$, $h^\Delta$, we define $h^\Delta \sim O(\Delta^p)$ for $p > 0$ as $\Delta \to 0$ if and only if

$$\limsup_{\Delta \to 0} \frac{h^\Delta}{\Delta^p} < \infty.$$

The following result establishes a property of the limiting behavior of $\beta_0^\Delta$ as $\Delta \to 0$.

**Lemma 4.** *We have*

$$\beta_0^\Delta \sim O(\Delta^{3/4}), \quad \text{as } \Delta \to 0.$$

*Proof.* From Lemma 3 we know

$$\begin{aligned} 0 < \beta_0^\Delta &= \frac{\sqrt{\Sigma_Z}}{\Sigma_0}\psi(\Sigma_0) \\ &= \frac{\sigma_z\sqrt{\Delta}}{\Sigma_0}\psi(\Sigma_0) \\ &\leq \frac{\sigma_z\sqrt{\Delta}}{\Sigma_0}[1-\rho]^{1/4}\sqrt{\Sigma_0} \\ &= \frac{\sigma_z}{\sqrt{\Sigma_0}}\sqrt{\Delta}[1-e^{-r\Delta}]^{1/4}. \end{aligned}$$

We have $[1-e^{-r\Delta}]^{1/4} \sim O(\Delta^{1/4})$ from which it follows that

$$0 \leq \limsup_{\Delta \to 0} \frac{\beta_0^\Delta}{\Delta^{3/4}} \leq \frac{\sigma_z}{\sqrt{\Sigma_0}} \limsup_{\Delta \to 0} \frac{\sqrt{\Delta}[1-e^{-r\Delta}]^{1/4}}{\Delta^{3/4}} < \infty,$$

which establishes the result. $\square$

Given these results, we now show that the main result: for all $\Delta > 0$ sufficiently small we have

$$\frac{\bar{J}_{d0} - \bar{J}_0}{\Delta} = \frac{-\frac{1}{2}\frac{(\beta_0^\Delta)^3\Sigma_0^2}{\sigma_z^2\Delta} + (1-e^{-r\Delta})c}{\Delta} > 0.$$

*Proof.* From Lemma 4 we know $\beta_0^\Delta \sim O(\Delta^{3/4})$. It follows that $(\beta_0^\Delta)^3 \sim O(\Delta^{9/4})$. Hence

$$\begin{aligned}\liminf_{\Delta\to 0}\left(-\frac{1}{2}\frac{(\beta_0^\Delta)^3\Sigma_0^2}{\sigma_z^2\Delta^2}\right) &= -\frac{1}{2}\frac{\Sigma_0^2}{\sigma_z^2}\limsup_{\Delta\to 0}\frac{(\beta_0^\Delta)^3}{\Delta^2}\\ &= -\frac{1}{2}\frac{\Sigma_0^2}{\sigma_z^2}\limsup_{\Delta\to 0}\frac{(\beta_0^\Delta)^3}{\Delta^{9/4}}\Delta^{1/4}\\ &= 0.\end{aligned}$$

Similarly since $\frac{1-e^{-r\Delta}}{\Delta} \to rc$, we have

$$\liminf_{\Delta\to 0}\frac{(1-e^{-r\Delta})c}{\Delta} = rc.$$

Combining the above two results yields

$$\liminf_{\Delta\to 0}\frac{\bar{J}_{d0}-\bar{J}_0}{\Delta} = rc > 0,$$

which establishes the result. □