Delayed Information Acquisition and Entry into New Trading Opportunities

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April 2019

Abstract

We model dynamic information acquisition and entry by a strategic trader. Instead of requiring the trader to commit before the market opens, we allow her to choose when to enter in response to public news. We characterize the unique equilibrium, in which entry is driven by public uncertainty and generically exhibits delay. The model provides novel implications for how the likelihood and timing of entry, and precision choice, depend on news volatility and the trading horizon. Our results shed light on why institutional investors delay entry into new opportunities, and how these dynamics vary across asset characteristics and market environments.

JEL: D82, D84, G12, G14

Keywords: dynamic information acquisition, delayed entry, slow moving capital

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1 Introduction

Informed capital often moves slowly and systematically across market conditions and asset classes. As Duffie (2010) and others highlight, the arrival of capital to trading opportunities can be delayed for extended periods of time, and this can lead to substantial and persistent dislocations between prices and underlying fundamentals. Moreover, in addition to specific trading opportunities, institutional investors also appear to wait to enter "new" asset classes (e.g., dot-com stocks in the 1990's, bitcoin in the 2010's) until market uncertainty and retail participation are sufficiently high.[1] Understanding these dynamics is important. Delayed and limited participation by sophisticated investors can lead to less informative prices which, in turn, decreases allocative efficiency (see Bond, Edmans, and Goldstein (2012) for a recent survey). The existing literature has proposed a number of explanations that rely on cognitive, institutional, or market frictions to understand slow-moving capital.[2] We propose a complementary channel whereby, even in the absence of such frictions, the strategic behavior of large, sophisticated investors can endogenously lead to delayed entry and information acquisition.

We begin with the observation that the value of information in financial markets changes over time and with economic conditions. Acquiring information and entering a new trading opportunity is more valuable when public uncertainty is high and speculative trading opportunities are more profitable. Since the entry and information acquisition decision is *irreversible*, an investor's optimal timing of entry resembles the optimal exercise of a real option (e.g., Dixit and Pindyck (1994)).[3] Intuitively, the investor trades off the net benefit of entering and accruing trading profits immediately against the benefit of waiting for entry / information to become more profitable as public news evolves. The resulting dynamics of entry exhibit delay and predictable variation with market conditions.

We develop a tractable, equilibrium model of strategic trading that emphasizes this "real options" feature of dynamic entry and information acquisition. Traditional models of strate-

[1] Griffin, Harris, Shu, and Topaloglu (2011) document that institutional holdings of dot-com stocks remained relatively low until late 1999, even though many of these firms came into existence in the mid-1990s. Similarly, trading volume and participation by institutional investors in bitcoin remained relatively low until 2017, even though the bitcoin network came into existence in January 2009 and the first exchange, BitcoinMarket.com, started operating in March 2010 - for example, see "Big investors yet to invest in bitcoin" by Alice Ross and Aliya Ram in the Financial Times, Oct 19, 2017 (https://www.ft.com/content/4c700f9a-b267-11e7-aa26-bb002965bce8).

[2] Examples include investor inattention (Duffie (2010)), search frictions in OTC markets (Duffie and Strulovici (2012)), funding frictions that restrict the ability of intermediaries to raise arbitrage capital (He and Krishnamurthy (2013)) and liquidity frictions that limit the ability of arbitrage capital to correct mispricing (Dow, Han, and Sangiorgi (2018)).

[3] Irreversible in the sense that the costs of entry / acquisition cannot be fully recouped if she later decides to leave the market.

gic trading assume that the entry / acquisition decision is "static," i.e., the investor commits to her information choices and entry decisions before the trading opportunity becomes available.[4] Instead, we allow the strategic trader to choose when to acquire private information and enter the market, in response to public news. We characterize the optimal acquisition and entry decision explicitly, and show that it exhibits delay relative to the naive "NPV" policy prescribed by a static model.[5] Moreover, we derive novel predictions that relate the likelihood and timing of entry to characteristics of the investment opportunity (e.g., uncertainty and trading horizon) and market conditions. Allowing for dynamic information acquisition delivers qualitatively distinct implications for the likelihood of information acquisition and the choice of information precision from those that arise in a static acquisition setting.

As described in Section 2, we begin with a continuous-time Kyle (1985) framework that builds on Back and Baruch (2004) and Caldentey and Stacchetti (2010). There is a single risky investment opportunity, traded by a risk-neutral, strategic investor and a mass of noise traders.[6] We introduce a publicly observable news process, which affects the market's uncertainty about the risky opportunity, evolves stochastically over time, and captures the time-varying value of information.[7] A risk-neutral market maker competitively sets the asset's price, conditional on the public news and aggregate order flow. The trading opportunity disappears at a random time when the risky payoff is publicly revealed.[8] In contrast to earlier work, we do not constrain the investor to make her information and entry choices before trading begins. Instead, we allow her to pay a cost at any point in time to privately acquire (noisy) information about the investment opportunity and enter the market.[9]

[4]Kyle (1985) and Back (1992) provide benchmarks for strategic trading models, while Back and Pedersen (1998) considers the effect of time-varying information arrival / acqusition in this setting with (implicit) ex-ante commitment to an arrival pattern.

[5]In a static setting, the investor enters if the expected trading profit upon entry exceeds the cost i.e., if the NPV (net present value) is greater than zero.

[6]We focus on the case of a single strategic investor for tractability. While an analysis of multiple investors is beyond the scope of this paper, we discuss possible outcomes in Section 7.

[7]One could instead consider settings without explicit public news but in which, e.g., noise trading volatility, explicit information costs, or prior uncertainty vary over time. Like ours, such a setting would generate time-varying value of information and lead to qualitatively similar results on entry behavior. However, settings with such features do not remain tractable once we introduce endogenous, dynamic entry. In light of this, one may interpret our model of public news as a reduced form for such alternative settings.

[8]The assumption of a random horizon is largely for tractability and is not qualitatively important for our primary results. What is key is that a random horizon induces the trader to discount future profits. We expect our results to carry over to settings that feature discounting for other reasons (e.g., if the trader has a subjective discount factor or the risk-free rate is nonzero).

[9]We treat acquisition and entry as a joint decision. As we discuss in Section 2.1 this is an economically reasonable assumption. Moreover, for simplicity, we assume that information comes in the form of a single, "lump" signal. As we discuss in Section 7, we expect our results to generalize to settings in which information instead (or additionally) flows in continuously after entry.

For concreteness, consider the decision of hedge fund deciding whether or not to enter into a risk arbitrage position following an M&A announcement. Information acquisition and participation are costly and irreversible: the fund must invest in research, analysis and relevant expertise before it takes a position. Importantly, the expected benefit from trading the trading opportunity varies over time and with market conditions. For instance, when the target and acquirer prices are stable (i.e., there is little public uncertainty about deal completion) and trading activity is limited, the value from entering the position is low. In contrast, high volatility and increased trading by less sophisticated traders (retail or noise traders) increase the value from participation. As a result, acquiring information and entering the market immediately need not be optimal; instead, the institution might prefer to wait until uncertainty about the investment opportunity is sufficiently high.

Given the tractability of our model, we explicitly characterize the investor's optimal entry and acquisition strategy in Section 3. We show that it follows a cutoff rule: she chooses to acquire information only when public uncertainty reaches a threshold. Furthermore, the optimal decision exhibits delay relative to a static "NPV" rule that prescribes entry as soon as the expected trading profit exceeds the cost of entry / acquisition. Consistent with the intuition from option exercise problems, the optimal threshold: (i) increases in the cost of acquisition / entry and the volatility of public news (both of which make waiting more attractive), and (ii) decreases in the volatility of noise trading and in the precision of the private signal (both of which make the trading opportunity more valuable).

We then characterize the economic implications of dynamic acquisition and entry in Sections 4 and 5. As a baseline, when the investor is constrained to make her entry / acquisition decision before trading begins, we show that: (i) when costs are sufficiently low, there is always information acquisition and entry, (ii) the likelihood of acquisition increases with the expected trading horizon, since the investor expects to exploit her informational advantage over a longer period of time, and (iii) from a given set of signals with varying precisions and correspondingly varying costs, the investor optimally chooses the signal with the highest "bang for buck" i.e., with the lowest cost-benefit ratio.

These "static" implications are qualitatively different from those in a dynamic setting, where the investor can choose when to acquire information and enter. First, for any given cost, the probability that acquisition / entry occurs is less than one as long as the volatility of the public news process is not too low. This is because higher news volatility increases the option value of waiting; moreover because the investment opportunity can disappear before the investor enters, more waiting reduces the likelihood of entry.

A second, key, prediction of our model that distinguishes it from static models of information acquisition (and alternative explanations for delay) is a non-monotonic relation between the likelihood of entry and the expected trading horizon. In particular, we show that the likelihood of acquisition / entry is hump-shaped in the trading horizon. When the payoff is expected to be revealed quickly (i.e., the horizon is short), the value from being informed is very low since there is little time over which to profit at the expense of noise traders. However, as the expected trading horizon increases, there are two offsetting effects. On the one hand, the value from being informed increases with the horizon since the trader expects her information advantage to last longer. On the other hand, the cost of waiting decreases with the horizon, since the likelihood that the payoff is revealed before acquisition is low. We find that initially the first effect dominates, while eventually the second one does. As a result, the trader is less likely to acquire information when the trading horizon is very long or very short.

Since informed entry is associated with a jump in return volatility and the resolution of payoff uncertainty is associated with a price jump in our model, this set of results implies a non-monotonic relation between price jumps and volatility jumps.[10] The model also predicts that entry by informed investors is more likely when diffusive volatility and trading volume is higher, and in assets or industries and/or trading opportunities with relatively high uncertainty (e.g., more intangible investment, higher growth opportunities, higher forecast dispersion). Importantly, the trading horizon results also distinguish our model from the canonical real options setting (e.g., McDonald and Siegel (1986); Dixit and Pindyck (1994)), where the probability of entry depends monotonically on the discount rate (the analogue of the expected trading horizon in our setting).[11]

Third, we show that the investor's choice of precisions does not depend only on the relative dollar costs across signals, since this ignores option to wait. For a fixed pair of signals, we show that the investor always prefers the high precision signal, irrespective of its relative cost, if either the news volatility is sufficiently high or the trading horizon is sufficiently long. In either case, the cost of waiting is relatively low, so it is never optimal to forgo the opportunity to acquire the more precise signal, even when it is relatively expensive to obtain. As such, the optimal choice of signal in a dynamic setting is not pinned down by just the relative precisions and costs of the signals, but also depends on the dynamics

[10] Importantly, the price jump in our model is driven by a substantive revelation of information, and as such, our predictions relate to relatively rare, "large" price movements driven by public information events.

[11] In the first generation of real options models (e.g, McDonald and Siegel (1986)), the value of the project, conditional on exercise, is an exogenous geometric brownian motion. In this case, the probability of entry increases in the discount rate since this reduces the option value of waiting. In generalizations (e.g., Dixit and Pindyck (1994), Ch. 6.1) when the investment project is an explicit set of discounted free cash-flows from selling products at exogenous price, the value of the project conditional on entry (i.e., NPV) is monotonically decreasing in discount rate. Moreover, this new effect dominates and the probability of entry shrinks in the discount rate (e.g., see Fig. 6.6 in Dixit and Pindyck (1994)).

of the investment opportunity. These results imply that, all else equal, more sophisticated investors tend to enter later than less sophisticated investors, and this effect is stronger for securities with higher uncertainty.

Unlike static models of information acquisition, our analysis also permits a characterization of the expected delay, conditional on entry.[12] Intuitively, the expected time of acquisition / entry is increasing in the cost of entry / acquisition, decreasing in the volatility of noise trading and in the precision of signals. More surprisingly, we find that the effect of news volatility and trading horizon are possibly non-monotonic, and depend on the cost of information acquisition. However, the conditional expected time to entry is decreasing in news volatility and increasing in trading horizon when either the cost is sufficiently high, news volatility is sufficiently high, or the trading horizon is sufficiently high.

Although stylized, our model provides guidance for understanding how participation by informed capital can vary across asset classes and economic environments. We discuss some of these empirical implications in Section 6. Our model is most directly applicable to settings where the the cost of entry and acquisition is "lumpy" or at least includes a fixed component.[13] At the level of individual securities, it is natural to study information acquisition and participation by institutional investors around discrete, value relevant news announcements (e.g., the announcements of mergers and acquisitions, changes in payout policy, new product launches or regulatory approvals, default decisions, unanticipated changes in leadership). More broadly, our model is applicable to entry by institutional investors into new trading opportunities and asset classes (e.g., dot-com stocks, cryptocurrencies). As we discuss further in Section 6, our model's predictions are broadly consistent with the empirically observed pattern of entry by institutional investors into new asset classes (like dot com stocks and bitcoin).

1.1 Related Literature

Following Grossman and Stiglitz (1980), a large literature has studied how investors choose to acquire information and participate in investment opportunities. A number of papers extend this basic setting to allow for dynamic trading (e.g., Mendelson and Tunca (2004), Avdis (2016)), to allow traders to condition their information acquisition decision on a public signal (e.g., Foster and Viswanathan (1993)), to allow traders to pre-commit to receiving signals

[12]Note that since the likelihood of acquisition / entry is less than one, the unconditional expected time of entry is infinite. Instead, we focus on the expected time of entry, conditional on there being entry.

[13]Since we want to better understand the nature of delay in participation by informed capital, the discreteness and irreversibility of the entry / acquisition decision are important. A setting where the investor is already in the market and trading, but only chooses the precision of her information (or the attention she allocates) would not exhibit delay without the introduction of additional frictions.

at particular dates (e.g., Back and Pedersen (1998), Holden and Subrahmanyam (2002)), to incorporate a time-cost of information (e.g., Kendall (2018), Dugast and Foucault (2018), and Huang and Yueshen (2018)), or to incorporate a sequence of one-period information acquisition decisions (e.g., Veldkamp (2006),Cai (2018)). However, in all of these papers, the information acquisition decision remains essentially static — investors make their information acquisition decision before the start of trade.

More recently, a number of papers consider the interaction of timing and information acquisition. Kendall (2018) studies whether or not investors wait for better quality information when there is no explicit monetary cost. In Dugast and Foucault (2018), investors can acquire a raw (less precise) signal which arrives early or a processed (more precise) signal which arrives later. In Huang and Yueshen (2018), investors can acquire both information and speed, which allows them to exploit their informational advantage in an earlier period. However, in all these papers, information acquisition decision is implicitly made prior to the start of trading.

Finally, our paper is related the recent literature that focuses on how information acquisition / production changes with the economic environment, and in particular, the business cycle (e.g., Kacperczyk, Van Nieuwerburgh, and Veldkamp (2016), Dow, Goldstein, and Guembel (2017), and Han (2019)). Our model provides a complementary analysis to these papers and shares a number of features (e.g., information is more valuable when uncertainty is high). However, while these papers focus on the variation in information production in competitive markets and its macroeconomic implications, our analysis focuses on the strategic behavior of large, institutional investors and the delay in entry / information acquisition. As such, our model generates a number of novel predictions about the dynamics of entry and information acquisitions that are absent in these papers. Moreover, to the best of our knowledge, our model is the first to model information acquisition by a strategic investor as a real options problem. Our analysis suggests that allowing for dynamic information acquisition has economically important consequences.

2 Model

Our framework is based on the continuous-time Kyle (1985) model with random horizon in Back and Baruch (2004) and Caldentey and Stacchetti (2010). Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is defined a 2-dimensional standard Brownian motion $\bar{W} = (W_\Delta, W_Z)$ with filtration $\mathcal{F}_t^W$, independent random variables ξ and T, and random variable S independent of all random variables except ξ. Let $\mathcal{F}_t$ denote the augmentation of the filtration $\sigma(\{\bar{W}_s\}_{\{0\leq s\leq t\}})$. Suppose that the random variable T is exponentially distributed with rate

r and that $\xi \in \{0, 1\}$ is binomial with probability $\alpha = \Pr(\xi = 1)$.

There are two assets: a risky asset (investment opportunity) and a risk-free asset with interest rate normalized to zero. The risky asset is of either high type (i.e., $\xi = 1$) or low type ($\xi = 0$). The type of the asset is not publicly known until a random terminal date T, when the asset pays off v. A high type asset pays off $v = H_T$ at the terminal date, while a low type asset pays off $v = L_T$. We model public beliefs about the conditional expected values of the two asset types in a reduced form way. In particular, we assume that L_t and H_t, the time-t conditional expected values of each type, follow exogenous processes, upon which we place more structure below.[14] Consequently, one can interpret the (publicly observable) process $\Delta_t \equiv H_t - L_t$ as news, since it reflects how the public's beliefs about the difference between the high and low type assets evolves over time.

To pin down ideas, consider the following examples. In the case of a new product introduction, investors face uncertainty about whether it is unsuccessful (i.e., $L_t \leq 0$) or successful (i.e., $H_t > 0$) and public news (e.g., adoption of the new product by consumers, sales figures, etc.) affects the expected value of the firm. . Similarly, in the case of fixed income securities, investors may be unsure about whether a specific bond will pay off at par (i.e., $H_t = 100$) or not (i.e., $L_t < 100$), and observable economic and market conditions determine the market's expectations of the loss given default (i.e., $\Delta_t = H_t - L_t$).

For notational clarity, and without loss of generality, we normalize the low payoff to $L_t \equiv 0$. For analytical tractability, we specify that Δ_t follows a geometric Brownian motion

$$\frac{d\Delta_t}{\Delta_t} = \sigma_\Delta \, dW_{\Delta t}, \tag{1}$$

where $\sigma_\Delta > 0$ and the initial value Δ_0 is a constant, which we normalize to $\Delta_0 \equiv 1$.[15] Compactly, the risky payoff at time T is therefore

$$v = L_T + \xi \Delta_T = \xi \Delta_T, \tag{2}$$

and the time-t conditional expected asset value, given knowledge of ξ and the history of Δ_t is $v_t = \xi \Delta_t$.

There is a single, risk-neutral strategic trader ("institutional investor") who can pay

[14]Because all market participants are risk-neutral, it is without loss of generality that we interpret the terminal payoff as a conditional expectation rather than the payoff itself. It is also straightforward to microfound particular dynamics for L_t and H_t by allowing market participants to observe noisy public signals about the value of each asset type. However, because our focus is on endogenous information acquisition about the asset type itself, rather than learning about a particular type from flow of public signals, we simply specify these processes exogenously.

[15]This assumption is not qualitatively important for our results, but ensures closed-form solutions to the optimal entry problem. We discuss our assumptions in more detail in Section 2.1 below.

a fixed cost $c > 0$ at any time τ to learn about the asset type and enter the market, and can optimally choose when to do so. If the investor acquires information and enters, she observes a noisy signal $S \in \{l, h\}$ about ξ that has a precision $q > 1/2$ i.e., $q = \Pr(S = h|v = H_T) = \Pr(S = l|v = L_T)$. This signal structure implies that the trader to has a conditional expectation given by:

$$\mathbb{E}[\xi|S] = \begin{cases} \frac{\alpha q}{\alpha q + (1-\alpha)(1-q)} \equiv \xi_h & \text{when } S = h \\ \frac{\alpha(1-q)}{\alpha(1-q)+(1-\alpha)q} \equiv \xi_l & \text{when } S = l \end{cases} \tag{3}$$

Let X_t denote the cumulative holdings of the trader, and suppose the initial position $X_0 = 0$. Further, suppose X_t is absolutely continuous and let $\theta(\cdot)$ be the trading rate (so $dX_t = \theta(\cdot)dt$).[16] There are noise traders who hold Z_t shares of the asset at time t, where

$$dZ_t = \sigma_Z dW_{Zt}, \tag{4}$$

with $\sigma_Z > 0$ a constant.

There is a competitive, risk neutral market maker who sets the price of the risky asset equal to the conditional expected payoff given the public information set. Let $\mathcal{F}_t^P$ denote the public information filtration, which we describe formally below. The price at time $t < T$ is given by

$$P_t = \mathbb{E}\left[v \middle| \mathcal{F}_t^P\right]. \tag{5}$$

Let $I_t = \mathbf{1}_{\{\tau \leq t\}}$ denote an indicator for whether the institutional investor has entered the market at time-t or before. Because the market maker observes the public signal Δ_t and order flow processes $Y_t = X_t + Z_t$, and the entry status of the investor, the public information filtration $\mathcal{F}_t^P$ is the augmentation of the filtration $\sigma(\{\Delta_t, Y_t, I_t\})$.[17] Let $\mathcal{T}$ denote the set of $\mathcal{F}_t^P$ stopping times. We require that the trader's information acquisition time $\tau \in \mathcal{T}$. That is, we require acquisition to depend only on public information up to that point. Let $\mathcal{F}_t^I$ denote the augmentation of the filtration $\sigma(\mathcal{F}_t^P \cup \sigma(S))$. Thus, $\mathcal{F}_t^I$ represents the institution's information set, post-entry. We require the trader's pre-entry trading strategy to be adapted to $\mathcal{F}_t^P$ and her post-acquisition strategy to be adapted to $\mathcal{F}_t^I$.

Finally, let π_t denote the market maker's conditional probability that the trader has observed a high signal, $S = h$. Note that zero and one are absorbing states for π_t. As such,

[16] Back (1992) shows that it is optimal for the trader to follow strategies of this form in a model in which she is exogenously informed.

[17] To reduce clutter, we abuse notation somewhat by using $\mathcal{F}_t^P$ to denote both the market maker's information set, which includes the acquisition indicator I_t in this case, as well as the institution's pre-acquisition (public) information set, which includes only the news process and order flow variables, and defines the admissible class of stopping times for acquisition.

following Back and Baruch (2004), we must rule out trading strategies that first drive the risky asset price to $\xi_l \Delta_t$ or $\xi_h \Delta_t$, incurring infinite losses, and then yield infinite profits by trading against a pricing rule that is unresponsive to order flows. To do so, we add to the existing smoothness and measurability restrictions on trading strategies a further condition which requires that the trading strategy for a trader informed of $S = h$ satisfies

$$\mathbb{E}\left[\int_0^T \Delta_u (1 - \pi_u) \theta_u^- \, du\right] < \infty, \tag{6}$$

and analogously for a trader informed of $S = l$,

$$\mathbb{E}\left[\int_0^T \Delta_u \pi_u \theta_u^+ \, du\right] < \infty. \tag{7}$$

A trading strategy that is smooth, satisfies the measurability restrictions, and satisfies (6) and (7) is *admissible*.

Our definition of equilibrium is standard, but modified to account for endogenous entry.

Definition 1. An equilibrium with pure strategy information acquisition is an entry time $\tau \in \mathcal{T}$ and *admissible* trading strategy θ for the trader, and a price process P_t such that, given the trader's strategy the price process satisfies (5) and, given the price process, the trading strategy and acquisition time maximize the ex-ante expected profit

$$\mathbb{E}\left[\int_0^T \theta(v_u - P_u) \, du\right]. \tag{8}$$

We focus on pure entry strategies. As we discuss below, we are in fact able to rule out the existence of equilibria with mixed-strategy entry.

2.1 Discussion of Assumptions

Note that the value of entering the market varies over time with variation in news Δ_t. To be clear, whether the asset type is high or low does not itself determine whether the value of the trading opportunity to the strategic trader is high or low. Rather, because short-selling is allowed, the value of the trading opportunity depends upon the extent to which the private signal gives her a more refined view of the asset value than the public. Moreover, this information is more valuable when the difference Δ_t in expected payoff between the two types is larger.

More generally, the specification of the public news process allows us to introduce stochastic volatility in a parsimonious and tractable manner, since the conditional variance of the payoff under the public information set prior to acquisition is

$$\text{var}[v|\Delta_t] = \alpha\,(1-\alpha)\,\Delta_t^2. \tag{9}$$

Without variation in public news ($\Delta_t \equiv 1$), the above setting reduces to the one analyzed by Back and Baruch (2004) but with endogenous, noisy information acquisition. In this case, however, the trader's acquisition decision is effectively static since the value of information is constant over time.[18] With a stochastic news process, the value of information evolves over time, which introduces dynamic considerations to the acquisition decision.

We expect alternative specifications that generate time-variation in uncertainty about fundamentals would generate similar predictions, although at the expense of tractability or a less natural economic interpretation.[19] Note that the assumption that Δ_t has zero drift is solely for simplicity and is, economically, without loss of generality. More generally, one could and replace Δ_t with $\mathbb{E}[\Delta_T|\mathcal{F}_t^P]$ in the pricing rule and trading strategy without qualitatively affecting the rest of the analysis. It is also straightforward to generalize to a general continuous, positive martingale for Δ, but at the expense of closed-form solutions to the optimal acquisition problem in most cases.

The assumption that information acquisition and entry occur simultaneously is made for simplicity, but it is economically reasonable. It is unlikely that a trader who has incurred the cost of acquiring private information about the risky opportunity will wait to begin trading, given that the opportunity may quickly disappear. Similarly, large investors are likely to be reluctant to enter and trade in a new market without some private informational advantage, especially since it is difficult to credibly convey to other market participants that one is uninformed.

We also assume that the acquisition / entry decision is detected by the market maker.[20] In general, entry by large investors into new markets is publicly scrutinized by the financial media. For instance, speculation about whether larger investment firms are setting up cryp-

[18] The acquisition decision would also be effectively static if uncertainty always decreased over time (e.g., if v was normally distributed, and the publicly observable signals were conditionally normal). In this case, uncertainty about v is highest at the beginning, and consequently, so is the value of acquisition and entry.

[19] Arguably, a more standard specification of the model would be one in which the value v is normally distributed with stochastic volatility (e.g., variance Σ_t). In order for this volatility to impact the acquisition decision, it must be publicly observable. However, this poses a difficulty: how does one interpret a setting in which the value of an asset is unobservable, but exhibits observable stochastic volatility? An alternative specification, in which there is a public signal with an error that exhibits stochastic volatility (e.g., $\Delta_t = v + \epsilon_t$, where ϵ_t exhibits stochastic volatility σ_t), necessitates the introduction of two state variables (i.e., the signal N_t and the conditional variance of v under the public information set, $\Sigma_{P,t}$), which limits tractability.

[20] We explore the implications of unobservable information acquisition in a companion paper — see Banerjee and Breon-Drish (2018).

tocurrency trading desks has been a recurring theme in recent news.[21] The addition of star traders, portfolio managers, and executives also garners significant media attention. Even if not covered by the popular press, participation by large traders is often known to other market participants. For instance, major broker-dealers that provide block trading services (or "upstairs trading desks") observe directly observe trading demand from institutional investors, and so can detect "entry" or increased participation. Similarly, prime brokers observe the cash and securities positions of their clients, and counter-parties in OTC derivative transactions disclose their interests to each other through ISDA agreements. Finally, many institutional investors are subject to regulatory reporting requirements, and disclosures about trading positions and capital adequacy can provide noisy information about an investor's trading strategies and private information.

3 Equilibrium

In this section we construct an overall equilibrium of the model by working backwards. First we characterize the equilibrium in the financial market given an entry time τ, and then we solve for optimal entry. We show that, generally, optimal entry exhibits delay. The entry decision by the investor resembles the exercise of a real option, and as such, the standard assumption that the investor makes a one-shot entry / information acquisition decision when the financial market opens is restrictive. Moreover, as we show in the next section, allowing for dynamic, endogenous entry has qualitatively novel implications for the likelihood of and timing of information acquisition and entry.

3.1 Financial market equilibrium

In the following result, we characterize the financial market equilibrium, conditional on an arbitrary acquisition time.

Proposition 1. *Fix an information acquisition time $\tau \in \mathcal{T}$. There exists an equilibrium in the trading game in which the price of the risky asset is given by $P_t = \Delta_t(\xi_h \pi_t + \xi_l(1 - \pi_t))$,*

[21] For example, see "Goldman Is Setting Up a Cryptocurrency Trading Desk" by Hugh Son, Dakin Campbell and Sonali Basak for Bloomberg.com on December 21, 2017 (https://www.bloomberg.com/news/articles/2017-12-21/goldman-is-said-to-be-building-a-cryptocurrency-trading-desk), and "BlackRock is evaluating cryptocurrencies, CEO Fink says" by Trevor Hunnicutt for Reuters.com on July 16, 2018 (https://www.reuters.com/article/us-blackrock-cryptocurrency/blackrock-is-evaluating-cryptocurrencies-ceo-fink-says-idUSKBN1K61MC).

where

$$\pi_t \equiv \mathbb{P}\left[S = h \mid \mathcal{F}_t^P\right] = \begin{cases} \hat{\alpha} & 0 \leq t < \tau \\ \Phi\left(\Phi^{-1}\left(\hat{\alpha}\right) e^{r(t-\tau)} + \sqrt{\frac{2r}{\sigma_Z^2}} \int_\tau^t e^{r(t-s)} dY_s\right) & \tau \leq t < T \end{cases} \tag{10}$$

where $\hat{\alpha} = \alpha q + (1-\alpha)(1-q)$ *is the prior probability that the trader will observe a high signal. Prior to information acquisition, the investor does not trade (i.e.,* $\theta^U \equiv 0$*), and conditional on entry / information acquisition, her strategy depends only on* π *and her signal, and is given by*

$$\theta^h(\pi) = \frac{\sigma_Z^2 \lambda(\pi)}{\pi}, \quad \text{and} \quad \theta^l(\pi) = -\frac{\sigma_Z^2 \lambda(\pi)}{1-\pi}, \tag{11}$$

where θ^i*,* $i \in \{U, h, l\}$*, denotes the trading strategy corresponding to prior to entry, informed of* $S = h$*, and informed of* $S = l$*. In this equilibrium, conditional on entry, the investor's value function is given by*

$$J^h(\pi_t, \Delta_t) = \Delta_t(\xi_h - \xi_l) \int_{\pi_t}^1 \frac{1-a}{\lambda(a)} da, \quad \text{and} \quad J^l(\pi_t, \Delta_t) = \Delta_t(\xi_h - \xi_l) \int_0^{\pi_t} \frac{a}{\lambda(a)} da, \tag{12}$$

where $\lambda(\pi) = \sqrt{\frac{2r}{\sigma_Z^2}} \phi\left(\Phi^{-1}(1-\pi)\right)$*.*

Our equilibrium characterization naturally extends the equilibrium in Back and Baruch (2004) to (i) accommodate the public news process Δ_t, (ii) account for the possibility that the investor is uninformed before the acquisition / entry time τ, and (iii) account for the noisy signal about ξ. Before entry, the investor does not trade,[22] and consequently, the order flow is uninformative and the market-maker does not update his beliefs from order flow. As a result, before τ the price is $P_t = (\hat{\alpha}\xi_h + (1-\hat{\alpha})\xi_l)\Delta_t = \alpha\Delta_t$ which is a geometric Brownian motion that evolves linearly with Δ_t. Conditional on information acquisition, the trader optimally trades according to θ^S characterized in the proposition. Since $\theta^h \neq \theta^l$, the order flow provides a noisy signal about S (and therefore ξ) to the market maker. The market maker's conditional beliefs about S, given by π_t, depend on the cumulative (weighted) order flow since the acquisition date (i.e., $\int_\tau^t e^{r(t-s)} dY_s$), and consequently, so does the price P_t.

[22] Under the posited price function, the pre-acquisition trading strategy is indeterminate. Any strategy that uses only public information earns zero expected profit under the public information set. Given such a trading strategy, it also remains optimal for the market maker to set $P_t = N_t\alpha$. Without loss of generality, we focus on the case in which the trader does not trade before time τ. In the presence of transaction costs, this would be the uniquely optimal strategy.

3.2 Optimal entry

Given the value function in Proposition 1, we characterize the optimal entry decision in the following result.

Proposition 2. *Given the financial market equilibrium in Proposition 1, there is a unique optimal entry strategy: the investor optimally enters the first time Δ_t hits the optimal entry boundary $\Delta^* = \frac{\beta}{\beta-1}\frac{c}{K}$ from below, where*

$$K = (\xi_h - \xi_l)\phi\left(\Phi^{-1}\left(1-\hat{\alpha}\right)\right)\sqrt{\frac{\sigma_Z^2}{2r}},\ \textit{and}\ \beta = \frac{1+\sqrt{1+8r/\sigma_\Delta^2}}{2}. \tag{13}$$

Moreover, the optimal acquisition boundary Δ^* increases in c and σ_Δ, decreases in σ_Z and q, is U-shaped in α (minimized at $\alpha = 0.5$), and is U-shaped in the expected trading horizon $1/r$.

In contrast, the standard approach in the literature restricts the strategic trader to make her information choices before trading begins. In this case, she follows a naive "NPV" rule — she only acquires information if the value from becoming informed is higher than the cost i.e., $\bar{J}\left(\Delta_0\right) \geq c$. As the following corollary highlights, the resulting information acquisition decision is effectively a static one.

Corollary 1. *If the investor is restricted to choosing acquisition and entry only at $t = 0$, she optimally acquires information if and only if $\Delta_0 \equiv 1 \geq \Delta_{NPV}$, where $\Delta_{NPV} = \frac{c}{K}$. Moreover, the optimal acquisition boundary Δ_{NPV} increases in c, decreases in σ_Z, is U-shaped in α (minimized at $\alpha = 0.5$), and decreases in the expected trading horizon (i.e., increases in r).*

As we show in the proof of Proposition 2, the expected profit immediately prior to entry at any date t (i.e., the value function the instant before ξ is observed) is given by

$$\bar{J}\left(\Delta_t\right) \equiv \mathbb{E}_t\left[\hat{\alpha}J^h\left(\hat{\alpha},\Delta_t\right) + \left(1-\hat{\alpha}\right)J^l\left(\hat{\alpha},\Delta_t\right)\right] = K\Delta_t. \tag{14}$$

Intuitively, the value function given information acquisition at date t (i.e., $K\Delta_t$) increases in the uncertainty about v. Specifically, note that $\bar{J}\left(\Delta_t\right)$ increases linearly in $\Delta_t = H_t - L_t$, the difference between the high and low payoff values. For a fixed prior uncertainty α about whether v is high or low, an increase in Δ_t leads to an increase in uncertainty about v. Similarly, the expected value from acquiring information also increases in the prior uncertainty about v (i.e., when α is closer to 0.5). The payoff from acquisition and entry is also higher when there is more noise trading (i.e., higher σ_Z), when the signal S is more precise (i.e., q is higher), and when the information advantage is expected to be longer lived (i.e., when r is smaller).

Given this expected payoff from information acquisition and entry, the optimal time to enter is characterized by the following optimal stopping problem:

$$J^U(\delta) \equiv \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[1_{\{\tau<T\}}(\bar{J}(\Delta_\tau) - c)\middle|\Delta_t = \delta\right] = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-r\tau}(K\Delta_\tau - c)^+ \middle|\Delta_t = \delta\right]. \qquad (15)$$

This problem is analogous to characterizing the optimal exercise time for a perpetual American call option.[23] Notably, the optimal entry decision exhibits delay: the investor does not enter the first instant that $K\Delta_t = c$, as would be implied by a naive, static NPV rule. The intuition for this effect is analogous to that for investment delay in a real options problem. At any point, the investor faces the following trade-off: she can enter now to begin exploiting her informational advantage against noise traders, or she can wait until uncertainty (i.e., Δ_t) is higher and her expected payoff from entry is larger. Since entry irreversibly sacrifices the ability to wait, it is optimal to enter only when doing so is sufficiently profitable to overcome this opportunity cost. Consistent with the intuition from real options problems, the option to wait is more valuable (and hence Δ^* is higher) when the volatility of the news process (i.e., σ_Δ) is higher.

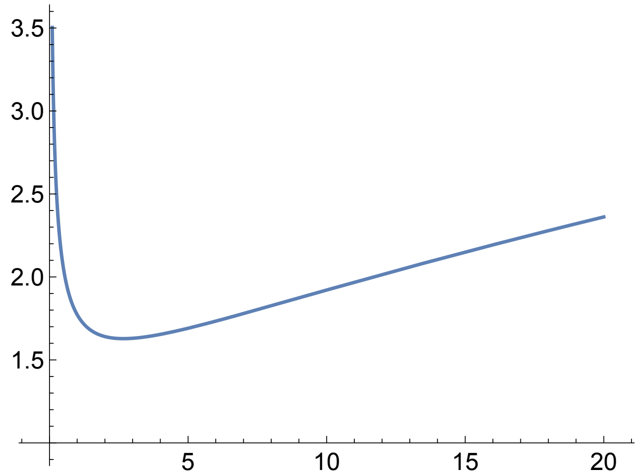
3.2.1 The effect of trading horizon on the optimal boundary

A key difference between the static entry boundary of Corollary 1 and the dynamic entry boundary of Proposition 2 is how they respond to the expected trading horizon (i.e., $1/r$). In the static case, an increase in the expected trading horizon (i.e., an decrease in r) leads to an decrease in the boundary Δ_{NPV}. This is intuitive: a longer trading horizon makes acquisition and entry more valuable since the trader can exploit her informational advantage over a longer window.

With dynamic entry, the trader also accounts for the cost of waiting to enter. Specifically, an increase in the trading horizon (i.e., an decrease in r) has two offsetting effects. First, as in the static case, it increases the value of acquisition, which pushes the boundary Δ^* downwards. Second, it decreases the cost of waiting since the likelihood that the value will be revealed before she can trade on the opportunity is lower. This pushes the boundary Δ^* upwards. As Figure 1 illustrates, this implies that the exercise boundary Δ^* is non-monotonic in the trading horizon ($1/r$). When the expected trading horizon is extremely short, the boundary is high because entry is not very valuable. Initially, the first effect dominates: an increase in the trading horizon leads to a decrease in the boundary Δ^*. However, eventually, the second effect over-comes the first — when the trading horizon is sufficiently high, further

[23] Hence, appealing to standard results, we establish that the optimal stopping time is a first hitting time for the Δ_t process and show that the given Δ^* is a solution to this problem.

Figure 1: Exercise Boundary Δ^* versus trading horizon
Unless otherwise specified, parameters are set to $\sigma_Z = \sigma_\Delta = 1$, $c = 0.25$ and $\alpha = 0.5$.



Δ^* versus trading horizon (i.e., $1/r$)

increases make waiting more attractive and so increase the boundary Δ^*.

4 The likelihood and timing of entry

In this section, we characterize the likelihood that the investor optimally acquires information and enters the market before the trading opportunity disappears. We show that allowing for dynamic entry / acquisition yield novel economic predictions that are not captured by standard models with static entry/acquisition. We then characterize how the expected time of entry, conditional on entry, depends on the underlying parameters of the model.

4.1 Likelihood of information acquisition and entry

The likelihood of entry depends on two forces. First, the cost of doing so may be too high relative to the value of acquiring it: given c, the trader might never find it optimal to exploit the investment opportunity if uncertainty does not become sufficiently high. Second, even if the cost of entry is not too high, the asset payoff may be revealed before the investor chooses to enter the market. The following result characterizes how these effects interact to determine the likelihood of entry.

Proposition 3. *Suppose acquisition does not occur immediately (i.e., $\Delta_0 = 1 < \Delta^*$). The probability that information is acquired is $\Pr(\tau < \infty) = \left(\frac{1}{\Delta^*}\right)^\beta$. The probability is decreasing in c, increasing in σ_Z and q, symmetric and hump-shaped in α (around $\frac{1}{2}$), and hump-shaped* *in the expected trading horizon $1/r$. If the cost c is sufficiently small (i.e., $c \leq K$), the probability is decreasing in σ_Δ; otherwise, it is hump-shaped in σ_Δ.*

Accounting for the possibility that the payoff is revealed before Δ_t hits Δ^* implies that information is not always acquired. More interestingly, it reveals novel comparative statics relative to those suggested in a static entry / acquisition setting.

First, the effect of changes in expected trading horizon (changes in $1/r$) is inherited from the effect of such changes on the optimal boundary Δ^*. When the trading horizon is short, the probability of acquisition and entry is low because, conditional on acquiring, the trader has little time to profit from her informational advantage. On the other hand, when the trading horizon is long, the probability of acquisition and entry is also low because in this case the cost of waiting is sufficiently low to offset the longer trading horizon conditional on acquiring. As such, the likelihood of entry is highest for intermediate trading horizons. In contrast, when the trader is restricted to choosing entry at $t = 0$, an increase in the expected trading horizon leads to an increase in the likelihood of entry.

Second, incorporating the possibility that the payoff is revealed before the trader acquires information also changes the effect of the volatility σ_Δ on the likelihood of acquisition. Increasing the volatility σ_Δ of Δ_t has two effects on the probability of acquisition: (i) it increases the acquisition boundary (i.e., Δ^* increases in σ_Δ), which tends to reduce the probability of acquisition, and (ii) fixing the boundary, it increases the likelihood that Δ_t will hit the boundary by any given time (i.e., Δ_t is more volatile), which tends to increase the probability of acquisition. The overall effect of σ_Δ therefore depends on the relative strength of these two forces.

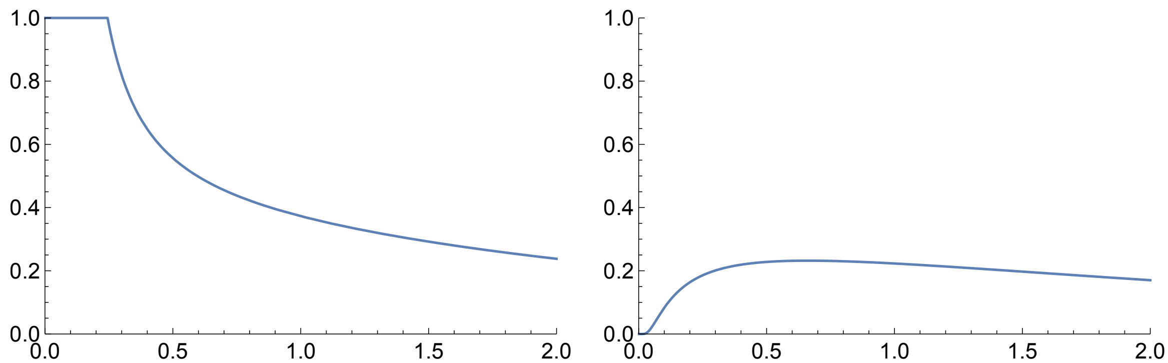
To gain some intuition for how the effect of σ_Δ depends on the acquisition cost c, note that the risky payoff is either $v = L_T = 0$ or

$$v = H_T = \Delta_T = e^{-\frac{1}{2}\sigma_\Delta^2 T + \sigma_\Delta W_T}.$$

As a result, the uncertainty about v, and consequently, the benefit of acquiring information depends on σ_Δ. When c is sufficiently low (i.e., $c \leq K$), uncertainty about v is already high enough for information acquisition and entry to be relatively valuable. Appealing to the analogy with an American call option, the option to acquire information starts in the money. In this case, an increase in volatility makes waiting more attractive (consequently, increasing the boundary Δ^*) and as a result, the likelihood of entry decreases.

In contrast, when the cost of entry c is relatively high (i.e., $c > K$), uncertainty about v is too low for entry to be valuable, i.e., the option starts out of the money. When volatility σ_Δ is low, the second effect initially dominates: an increase in σ_Δ increases the likelihood

Figure 2: Probability that information is acquired $\Pr(\tau < \infty)$ versus σ_Δ.
Unless otherwise specified, parameters are set to $\sigma_Z = 1$, $c = 0.25$ $r = 1.5$, $\alpha = 0.5$.



(a) $c = 0.2$ (i.e., $c < K$)

(b) $c = 0.25$ (i.e., $c > K$)

that Δ_t will hit the boundary, and so increases the likelihood of entry. However, once σ_Δ is sufficiently high, the effect on the boundary overwhelms this effect: further increases in σ_Δ make waiting more attractive and decrease the likelihood of entry.

Figure 2 presents an example of this non-monotonic effect of σ_Δ on the probability of information acquisition. In panel (a), the cost of acquisition and entry is sufficiently low (i.e., $c \leq K$) so that the probability of information acquisition is decreasing in σ_Δ. In panel (b), the cost is relatively high (i.e., $c > K$) so that the probability of information acquisition initially increases and then decreases in σ_Δ.

4.2 Expected time of entry

Using the distribution of τ derived in the proof of Proposition 3, we characterize the expected time of information acquisition in the following result.

Proposition 4. *Suppose acquisition does not occur immediately (i.e., $\Delta_0 < \Delta^*$). All unconditional moments of τ are infinite. The expected time of entry, conditional on entry occurring, is*

$$\mathbb{E}[\tau|\tau < \infty] = \frac{2\log(\Delta^*)}{\sigma_\Delta^2\sqrt{1 + \frac{8r}{\sigma_\Delta^2}}}. \tag{16}$$

Moreover, $\mathbb{E}[\tau|\tau < \infty]$ is increasing in c, decreasing in σ_Z and q, U-shaped in α. When the cost c is sufficiently large, the conditional expected time is decreasing in σ_Δ and increasing in expected trading horizon $(1/r)$; otherwise, it may be non-monotonic in either parameter.

Since information acquisition / entry does not always occur, the *unconditional* moments of τ are infinite. However, conditional on entry, the expected time of entry is characterized by expression (16). To gain some intuition, consider the numerator and denominator of the expression in (16) separately. Intuitively, an increase in the acquisition boundary Δ^* implies there is more delay in entry / acquisition since it takes longer for Δ_t to cross Δ^* — this is reflected by the numerator. Moreover, for a fixed Δ^* and conditional on entry, a higher volatility (i.e., higher σ_Δ) implies entry must have occurred faster (on average) — a more volatile Δ_t process would take less time to cross the threshold. Similarly, conditional on entry having happened, a shorter trading horizon (i.e., higher r) implies entry must have occurred earlier since it occurred before the value is publicly revealed. These conditional effects are reflected in the denominator of (16).

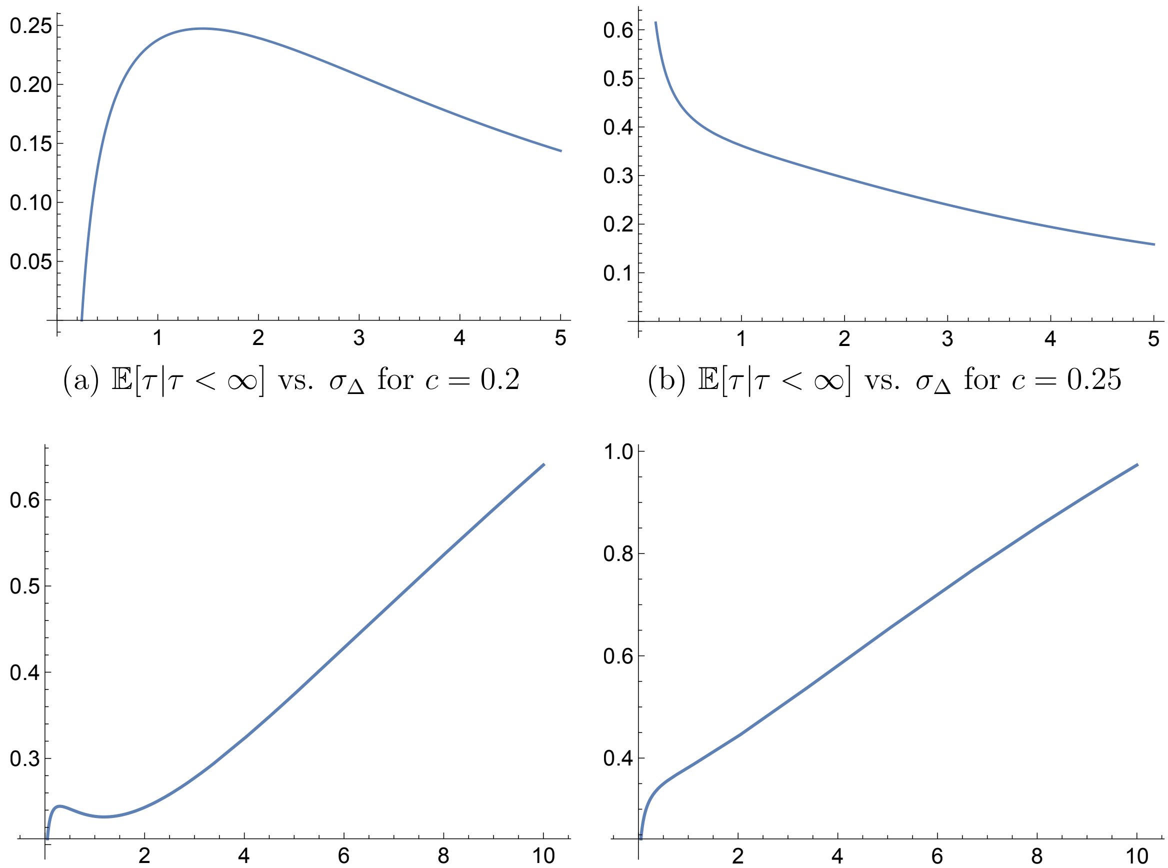
The comparative statics with respect to c, σ_Z, q and α are intuitive and inherited from the dependence on Δ^*. Recall that delaying acquisition / entry becomes more attractive (i.e., Δ^* increases) with a decrease in noise trading (σ_Z), signal precision (q), or prior uncertainty about v (i.e., $\alpha(1-\alpha)$), and with an increase in the cost of acquisition / entry (c). As a result, the conditional expected time to acquisition increases with each of these changes.

As Figure 3 illustrates, the effects of volatility (σ_Δ) and trading horizon ($1/r$) are more nuanced since they impact both the numerator and the denominator of (16). When the cost c is sufficiently large, the denominator channel dominates: the conditional expected time of entry decreases with volatility σ_Δ and increases with the trading horizon $1/r$ — see panels (b) and (d), respectively. However, when the cost c is relatively small, the effects through the Δ^* term in the numerator interact with those from through the denominator to generate non-monotonic effects, as illustrated in panels (a) and (c). Notably, when both the cost of entry (c) and volatility (σ_Δ) are sufficiently small, the option to enter is deep in the money and immediate entry is optimal — in this case, the conditional expected time is zero. For low levels of σ_Δ, an increase in volatility can lead to a higher acquisition boundary, and consequently, a higher conditional expected time (e.g., see panel (a)). Similarly, when the trading horizon is short (i.e., r is large), the conditional expected time initially increases, then decreases and then increases in $1/r$.

5 Precision choice and entry dynamics

In this section, we allow the institutional investor to choose not only the timing of information acquisition and entry, but also the precision of her signal. Specifically, suppose at the time of entry, the investor can either acquire a signal with high precision q_h at cost c_h or acquire one with a with low precision q_l at cost c_l, where $\frac{1}{2} < q_l < q_h \leq 1$ and $0 < c_l \leq c_h$. As such, her decision depends not only on the tradeoff between cost and precision, but also on the

Figure 3: Conditional expected time of entry $\mathbb{E}[\tau|\tau < \infty]$ vs. σ_Δ and $1/r$.
Unless otherwise specified, parameters are set to $\sigma_Z = 1$, $\sigma_\Delta = 1$, $r = 1.5$, $\alpha = 0.5$.



(a) $\mathbb{E}[\tau|\tau < \infty]$ vs. σ_Δ for $c = 0.2$

(b) $\mathbb{E}[\tau|\tau < \infty]$ vs. σ_Δ for $c = 0.25$

(c) $\mathbb{E}[\tau|\tau < \infty]$ vs. $1/r$ for $c = 0.2$

(d) $\mathbb{E}[\tau|\tau < \infty]$ vs. $1/r$ for $c = 0.25$

timing of entry: she can acquire the cheaper, less precise signal but begin trading earlier or wait to acquire the more informative, more expensive signal.

As before, we assume that the market maker can detect the timing of entry. To ensure tractability, we further assume that the market maker can detect the precision choice of the strategic trader. As discussed earlier, entry into a new market involves hiring of analysts and traders. The reputations and track records of such individuals are likely to convey information (albeit noisy) about their quality. The assumption ensures that the financial market equilibrium in the current setting is given by the characterization in Proposition 1, given the optimal choice of precision $q^* \in \{q_l, q_h\}$. In what follows, let K_l and K_h denote the the value function coefficient characterized by (13) that correspond to the precision choice of q_l and q_h, respectively. We begin with an observation about the static acquisition / entry setting.

Lemma 1. *If the investor is restricted to choosing acquisition and entry only at $t = 0$, the* *trader acquires the high precision signal iff* $\frac{c_h}{K_h} \leq \min\left\{1, \frac{c_l}{K_l}\right\}$ *and acquires the low precision signal iff* $\frac{c_l}{K_l} \leq \min\left\{1, \frac{c_h}{K_h}\right\}$.

When the acquisition / entry is a static decision, the investor optimally chooses the signal with the highest "bang for buck," or equivalently, the lowest cost-benefit ratio. In contrast, the following result characterizes the optimal choice of the investor with dynamic acquisition and entry.

Proposition 5. *Let* $\bar{\Delta} \equiv \frac{c_h - c_l}{K_h - K_l}$, $\Delta_l^* = \frac{\beta}{\beta-1}\frac{c_l}{K_l}$ *and* $\Delta_h^* = \frac{\beta}{\beta-1}\frac{c_h}{K_h}$, *where* β, K_h *and* K_l *are characterized as in* (13). *Given the financial market equilibrium in Proposition 1 , there is a unique optimal entry strategy: the investor optimally enters the first time* Δ_t *hits the optimal entry boundary* Δ^* *from below, where*

$$\Delta^* = \begin{cases} \Delta_l^* & \textit{if } \Delta_l^* < \bar{\Delta} \textit{ and } K_h^{\beta} c_h^{1-\beta} \leq K_l^{\beta} c_l^{1-\beta} \\ \Delta_h^* & \textit{otherwise} \end{cases}. \tag{17}$$

The equilibrium characterized by Proposition 5 is intuitive. As in the equilibrium of the benchmark model, the optimal acquisition time is given by the first time Δ_t hits the optimal acquisition boundary Δ^*. Moreover, the optimal boundary corresponds to either the boundary when only the high precision signal is available (i.e., $\Delta_h = \frac{\beta}{\beta-1}\frac{c_h}{K_h}$), or when only the low precision signal is available (i.e., $\Delta_l = \frac{\beta}{\beta-1}\frac{c_l}{K_l}$). Finally, note that $\bar{\Delta}$ is the threshold at which the investor is indifferent between acquiring the high precision and low precision signals i.e.,

$$K_h\bar{\Delta} - c_h = K_l\bar{\Delta} - c_l. \tag{18}$$

It is straightforward to show that if $\frac{c_h}{K_h} < \frac{c_l}{K_l}$ then the conditions for a low-precision signal are never met. Hence, the investor prefers the low precision signal only when (i) the cost-benefit ratio for the low precision signal is better (i.e., $\frac{c_l}{K_l} < \frac{c_h}{K_h}$), and (ii) the low precision leads to a sufficiently high value function that waiting and later acquiring a high-precision signal is not optimal. In contrast, the investor prefers the high precision signal if either the cost-benefit ratio for the high precision signal is better i.e., $\frac{c_l}{K_l} \geq \frac{c_h}{K_h}$, or if the the effective cost of waiting to hit Δ_h^* is not too high.

Importantly, unlike the case where acquisition / entry is a static decision, the optimal choice of precision does not only depend on the relative costs (i.e., c_l vs c_h) and precisions of the two signals (i.e., q_l and q_h, via K_l and K_h). Instead, it also depends on the volatility of the news process (i.e., σ_Δ) and the expected trading horizon (i.e., $1/r$) though their effect on β and therefore Δ_l^*. In fact, this leads to the following striking result.

Corollary 2. *Fixing any pair of precisions and costs (i.e., for fixed* $\{q_l, c_l\}$ *and* $\{q_h, c_h\}$*), the investor always prefers the high precision signal if either the volatility* σ_Δ *of the news process is sufficiently high, or if the expected trading horizon* $1/r$ *is sufficiently long (i.e.,* r *is sufficiently small).*

The result implies that when news volatility is sufficiently high or the trading horizon is sufficiently long, the investor optimally chooses the high precision signal, irrespective of its relative cost-benefit ratio. If news volatility is sufficiently high, or the trading horizon is sufficiently long, the cost of waiting is relatively low. Since acquiring a low-precision signal irreversibly forgoes the ability to acquire a high-precision signal, it is never optimal to acquire a low-precision signal, regardless of its direct cost-benefit ratio. Furthermore, under these conditions, it follows immediately that institutions with access to more precise information will *always* wait longer to enter than otherwise-equivalent institutions with access only to low-precision information.

Corollary 3. *Fixing any pair of precisions and costs (i.e., for fixed* $\{q_l, c_l\}$ *and* $\{q_h, c_h\}$*), an investor who can choose between the two signals always waits longer to enter than an otherwise identical investor who has access only to the low precision signal if either the volatility* σ_Δ *of the news process is sufficiently high, or if the expected trading horizon* $1/r$ *is sufficiently long (i.e.,* r *is sufficiently small).*

The result suggests that, all else equal, under the conditions in the Corollary, larger and more sophisticated institutions that have access to better expertise (and hence more "precise signals") are likely to wait longer to enter new investment opportunities. This is a unique prediction of the dynamic model and would not necessarily obtain in a static model (e.g., if the naive NPV of the low-precision signal is higher than that of the high-precision signal). In the next section, we provide more guidance on how an econometrician might approach this, and other, predictions of the model.

6 Empirical Implications

A key challenge in testing the empirical predictions of our model is that entry by large investors may not be observable by an econometrician, even when it is detectable by other market participants.[24] One approach would be to directly proxy for the entry and participation of large investors using regulatory filings. For instance, Schedule 13D filings can be

[24]Importantly, note that our model and consequently, the implications we discuss below, presume that other market participants can detect entry / increased participation by large investors. As we discussed in Section 2.1, this is often the case: market makers, prime brokers, and OTC counter-parties observe trading demand from institutional investors.

used to identify trading by large investors who have acquired more than 5% of any class of securities of a publicly traded company (e.g., as in Collin-Dufresne and Fos (2015) and Brav, Jiang, Partnoy, and Thomas (2008)). Similarly, changes in the panel of quarterly Schedule 13F filings (required for any institution with at least $100m under management) can be used to estimate large position changes associated with information acquisition by large institutional investors such as hedge funds (e.g., Griffin and Xu (2009), Agarwal, Jiang, Tang, and Yang (2013)), and one can use Thomson and CRSP data to do the same for mutual funds (e.g., Wermers (2000)). Further, by distinguishing between initiation of new positions and changes in existing holdings, such filings allow one to, at least partially, separate entry from changes in trading intensity. Recently, papers have proposed more direct measures of public information demand from institutional investors, including Bloomberg queries (Ben-Rephael, Da, and Israelsen (2017)) and online requests to the EDGAR system (e.g., DeHaan, Shevlin, and Thornock (2015), Loughran and McDonald (2017)). To the extent that increased demand for such public information is accompanied by acquisition of private information (or essentially represents private information generation via more effective processing / interpretation of public releases), these measures provide a noisy proxy for increases in information acquisition. Finally, one could use institutional investor participation in secondary market platforms like SharesPost and Nasdaq Private Market, which allow qualified investors to transact in shares of private companies, as a proxy for entry into new investment opportunities.[25]

Alternatively, one can take the opposite approach and use the model to make inferences about the entry of informed investors by using the model's implications for return dynamics. Specifically, Proposition 1 implies that entry by an informed trader is associated with (i) a jump in return volatility, and (ii) a jump in price impact, or estimated Kyle's lambda. As such, abrupt increases in estimated return volatility and price impact imply an increased likelihood of entry by informed investors.

One could generate a sample of events within which to test the model's predictions by matching these instances of entry and / or increased intensity of informed trading to subsequent public announcements of discrete, value relevant news (e.g., announcements of mergers and acquisitions, surprising changes in payout policy, announcement of new products or market entry, regulatory approvals of new drugs).[26] Broadly, the model generates three types of predictions:

[25]See `https://www.sharespost.com` and `https://www.nasdaqprivatemarket.com`, respectively, for more information.

[26]Notably, our model's predictions relate to relatively "large", relatively infrequent, information driven, price jumps, but not to higher-frequency (intraday) , "normal" jumps that may arise due to microstructure effects.

- Predictions about trading horizons: A distinguishing prediction of our dynamic model relative to similar static models is the hump-shaped relation between the likelihood of entry / information acquisition and the expected trading horizon (see Proposition 3). Moreover, in the model, the resolution of payoff uncertainty (at time T) is associated with a jump in the price level. As such, the model implies that the frequency of price jumps due to public information revelation and volatility jumps exhibit a non-monotonic relation: the frequency of volatility jumps (entry by informed traders) should be highest for securities with an intermediate frequency of price jumps, but lower for securities with more or less frequent price jumps. One could also test the model's predictions by studying how entry and trading varies for announcement types that differ in frequency, e.g, M&A announcements (relatively rare) vs. substantive capital structure or dividend announcements (more frequent) vs. new product introductions or other product market announcements (even more frequent).

- Predictions about likelihood and timing of entry: The model predicts that higher (ex-ante) public uncertainty and higher volatility of noise trading should correspond to higher likelihood of entry and informed trading. In the time-series, this suggests that for a given security, periods of higher diffusive volatility and higher trading volume should be followed by jumps in return volatility, and by periods of higher price impact and volatility of volatility. In the cross-section, stocks in industries with relatively higher payoff uncertainty (e.g., more intangible investment, higher growth opportunities, higher forecast dispersion) and/or more noise trading (e.g, retail trading) should exhibit higher likelihood of informed investor participation and entry. The model makes similar predictions about the expected delay, conditional on entry, which could be evaluated by examining cross-sectional differences in entry timing, conditional on observing entry in 13D / 13F filings.

- Predictions about types of informed traders: The results of Section 5 imply systematic differences in how entry timing depends on the sophistication of investors. Specifically, the model predicts that for securities with high uncertainty, more sophisticated investors (with access to more precise signals) tend to enter later than less sophisticated investors, and this effect is stronger for securities with higher uncertainty. These implications are specific to our model as they are driven by the option value of waiting that is present only in a dynamic setting. One could test these predictions using a panel of inferred entry / informed trading based on Schedule 13D / 13F filings. For a given security, the model predicts that less sophisticated institutional investors (e.g., small, passive mutual funds) participate earlier than more sophisticated investors (e.g., large, active mutual funds and hedge funds). Furthermore, in such settings, the model suggests a positive relationship between entry timing and average gross trading profit for the relevant holding(s).

In addition to entry behavior around value-relevant corporate events, our model is also useful in understanding entry dynamics in new asset classes, and how this varies with asset and market characteristics. For a recent example, bitcoin/cryptocurrencies are characterized by (i) an evidently long trading horizon[27], (ii) a sharp increase in prices and uncertainty in 2017, and (iii) increased demand from uninformed, retail investors (noise traders), especially in the second half of 2017. As such, our model is consistent with low institutional entry until 2017 when, as a result of higher volatility and more noise trading, the value of market participation increased sharply. Moreover, our model predicts that when the trading horizon is sufficiently long, investors choose the highest precision signals available to them, and consequently, wait longer. This suggests that more sophisticated investors remain out of the market for longer, which appears consistent with the decisions of many large asset management firms.[28] Finally, relative to cryptocurrencies, entry into tech stocks in the 90s featured less delay and was more widespread. This is consistent with our model's predictions since (i) market uncertainty was arguably lower for tech stocks, (ii) the trading horizon for tech stocks was neither extremely short nor extremely long.

It is important to note that our model's predictions are distinct from those in Collin-Dufresne and Fos (2016), who suggest that *conditional on entry*, there may be a negative relationship between measured price impact and informed trading intensity if noise trading volatility is stochastic and the strategic trader can time her trades during times of high noise volatility. Our results relating noise trading intensity and subsequent price impact are driven by increased entry by informed traders. Our results are complementary to theirs, and operate and a lower frequency: while their focus is on daily (or higher frequency) variation in measured price impact and the incidence of strategic trading, our claim is that abrupt changes in measured price impact are associated with market maker detection of entry by large informed investors.

It is also important to note that in Collin-Dufresne and Fos (2016) (as well as most other continuous-time Kyle-style models, including ours), conditional on entry, the trader's optimal strategy is not uniquely defined. Rather, as first identified by Back (1992), in equilibrium,

[27] Despite the fact that bitcoin has been traded for almost a decade, the true value of the currency remains uncertain and hotly debated. For a recent survey, see "What 12 major analysts from banks like Goldman, JPMorgan, and Morgan Stanley think of bitcoin," by Will Martin in Business Insider, on January 18, 2018 (http://www.businessinsider.com/bitcoin-round-up-wall-street-cryptocurrencies-bull-bear-market-2018-1).

[28] See "Bitcoin extends gains as BlackRock looks into crypto and blockchain" by Ryan Browne and Fred Imbert for CNBC on July 16, 2018 (https://www.cnbc.com/2018/07/16/bitcoin-jumps-after-report-says-blackrock-exploring-cryptocurrencies.html).

the trader is indifferent between following her posited equilibrium strategy or waiting and not trading for an arbitrary interval of time and then following her prescribed strategy from that point forward.[29] This makes it difficult to draw sharp conclusions about the relation between the trading strategy (conditional on entry) and other objects of interest.

7 Conclusions

When do traders choose to enter into new markets and exploit new investment opportunities? To study this question, we develop a strategic trading model in which a trader can endogenously choose *when* to acquire information and enter in response to the evolution of a public signal. While a number of papers have studied entry/information acquisition in financial markets, this work typically assumes that investors make a one-shot decision at the time that the market opens. As such, these models are not well-suited for studying the optimal timing of information acquisition and entry.

We show that when a trader can optimally choose when to enter, there is generally delay beyond what is prescribed by a naive "NPV" rule. Furthermore, allowing for dynamic entry/acquisition provides qualitatively novel economic implications relative to a model of "static entry". In particular, we derive new predictions for how the likelihood and timing of entry, as well as optimal precision choice, depend on news volatility and the expected trading horizon.

More broadly, our analysis suggests that key features of the standard strategic trading framework may be difficult to reconcile with the dynamic entry/acquisition decisions of large traders. Exploring the robustness of our results to various assumptions is a natural next step. While entry by large institutions into new opportunities is likely to be observed by other market participants, information acquisition by existing investors may be more difficult to detect. In related work Banerjee and Breon-Drish (2018), we study an alternative setting in which an investor's information acquisition is not detected by the market maker, but must instead be filtered from order flow. We show that unobservable acquisition can lead to market breakdown.

Another important extension would be to consider competition among multiple strategic traders. When the number of strategic investors becomes arbitrarily large, we conjecture that competitive forces eliminate delay — investors should acquire information and enter

[29] More precisely, in a fixed-horizon model the trader is indifferent among strategies that drive the price to the asset value at the terminal date. In a a random-horizon model, the trader is indifferent among strategies that drive the price to the asset value in the limit when trading continues an arbitrarily long period of time (i.e., when the random terminal date does not arrive until arbitrarily long in the future). In both cases, optimality is essentially equivalent to not leaving any "money on the table" at the end of the game.

the market as soon as the "naive" NPV of doing so is positive. The case of imperfect competition among multiple, strategic traders is technically challenging, but we expect the results from such a model to be qualitatively similar to those in this paper as long as private information is not perfectly correlated across traders (see Foster and Viswanathan (1993) and Back, Cao, and Willard (2000) for the effect of imperfect competition in strategic trading settings).

We assume that the information acquired by the investor is "lumpy" for tractability. We expect results on delay to be qualitatively similar to settings in which entry is associated with a flow of future private information instead of (or in addition to) a single signal. In fact, Back and Pedersen (1998) show that in a continuous-time Kyle setting, the strategic trader is indifferent among information arrival processes that provide the same total amount of information, so there is a sense in which our model is isomorphic to such a setting. What is important for delay is that the initial cost of information acquisition / entry has a fixed component, not that the information observed is discrete. However, it would also be interesting to study settings in which the trader makes more than one entry / acquisition decision (e.g., continuously optimizes flow of private information, or a sequence of "lumpy" signals), and to understand the effect of endogenous public news (e.g., in the form of strategic disclosure by firms or regulators). We hope to explore these extensions in future work.

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A Proofs

Proof of Proposition 1. To establish the equilibrium in the Proposition, we need to show: (i) the proposed price function is rational, and (ii) the informed trader's strategy is optimal. Fix any $\tau \in \mathcal{T}$.

Rationality of pricing function

Consider the set $\{t : t < \tau\}$ on which the trader has not acquired information. Then, because $\{\Delta_t\}$, $\{Z_t\}$ and ξ are independent, and under the proposed trading strategy $Y_t = Z_t$ for $t < \tau$, it is immediate that

$$\mathbb{E}[\xi\Delta_T|\mathcal{F}_t^P] = \mathbb{E}[\xi|\mathcal{F}_t^P]\mathbb{E}[\Delta_T|\mathcal{F}_t^P] = \alpha\mathbb{E}[\Delta_T|\mathcal{F}_t^P].$$

Since T is almost surely finite and is independent of the process Δ_t we have $\mathbb{E}[\Delta_T|\mathcal{F}_t^P] = \Delta_t$, and so $\mathbb{E}[\xi\Delta_T|\mathcal{F}_t^P] = \alpha\Delta_t$.

Now, consider the set $\{t : \tau \le t < T\}$ on which the trader has entered and the asset payoff has not yet occurred. Up to the addition of the news process and the noisy signal, the problem now resembles that considered in Back and Baruch (2004), and we can adapt the proof offered there. Specifically, consider the updating rule from Back and Baruch (2004), adapted for the fact that the signal is acquired at time τ,

$$d\pi_t = \lambda(\pi)dY_t, \quad \pi_\tau = \hat{\alpha},$$

where $\lambda(\pi)$ is given in the statement of the Proposition. (Later we will show that this pricing rule can be written in the explicit form in eq. (10).) Note that the proposed trading strategy depends only on S and π, the process π depends only on the order flow, and $\{\Delta_t\}$ is independent of S and $\{Z_t\}$, so $(S, \{\pi_t\})$ is conditionally independent of $\{\Delta_t\}$, and therefore

$$\mathbb{E}[\xi\Delta_T|\mathcal{F}_t^P] = \mathbb{E}[\xi|\mathcal{F}_t^P]\mathbb{E}[\Delta_T|\mathcal{F}_t^P] = \mathbb{P}[\xi = 1|\mathcal{F}_t^P]\mathbb{E}[\Delta_T|\mathcal{F}_t^P] = \mathbb{P}[\xi = 1|\{Y_s\}_{s\le t}]\Delta_t = (\xi_h\pi_t + \xi_l(1-\pi_t))\Delta_t,$$

where the next-to-last equality follows since $\mathbb{E}[\Delta_T|\mathcal{F}_t^P] = \Delta_t$. Furthermore, since $Y_t = Z_t$ for $t < \tau$ under the proposed trading strategy and S is independent of $\{Z_t\}$ it follows that $\mathbb{P}[\xi = 1|\{Y_s\}_{s\le t}] = \mathbb{P}[\xi = 1|\{Y_s\}_{\tau\le s\le t}]$.

Recall that as of time τ, the informed trader begins trading according to the strategy $\theta^S(\pi)$ and the order flow becomes informative. The market maker's conditional expectation is simply equal to her prior $\hat{\alpha}$ since before this time only noise traders have been active. It follows that starting at time τ the market maker's filtering problem becomes identical to that of the market maker in Back and Baruch (2004), modified to account for the fact that she is filtering one of two signal realizations rather than ξ itself. Hence, their Theorem 1 implies that for $t \geq \tau$ the pricing rule

$$d\pi_t = \lambda(\pi) dY_t, \qquad \pi_\tau = \hat{\alpha},$$

satisfies $\pi_t = \mathbb{P}[\xi = 1 | \{Y_s\}_{s \geq \tau}]$.

To complete the proof of the rationality of the proposed price, it suffices to show that the explicit form of $\pi(\cdot)$ for $\tau \leq t < T$ in eq. (10) satisfies $d\pi_t = \lambda(\pi) dY_t$. Applying Ito's Lemma to the function $f(\pi) = \sqrt{\frac{\sigma_Z^2}{2r}} \Phi^{-1}(\pi)$ to the above process for π_t gives

$$\begin{aligned} df(\pi_t) &= \frac{1}{2} \sigma_Z^2 \lambda^2(\pi_t) \frac{\frac{2r}{\sigma_Z^2} f(\pi_t)}{\lambda^2(\pi_t)} \, dt + \frac{1}{\lambda(\pi_t)} \lambda(\pi_t) dY_t \\ &= r f(\pi_t) \, dt + dY_t. \end{aligned}$$

Now applying Ito's lemma to the function $e^{-rt} f(\pi_t)$ and integrating allows one to express

$$f(\pi_t) = f(\pi_\tau) e^{r(t-\tau)} + \int_\tau^t e^{r(t-s)} dY_s.$$

Note that $f(\pi_\tau) = \sqrt{\frac{\sigma_Z^2}{2r}} \Phi^{-1}(\hat{\alpha})$, so returning to the explicit form of the function $f(\pi)$ and inverting it follows that

$$\pi_t = \Phi \left(\Phi^{-1} (\hat{\alpha}) \, e^{r(t-\tau)} + \sqrt{\frac{2r}{\sigma_Z^2}} \int_\tau^t e^{r(t-s)} dY_s \right).$$

Optimality of trading strategy

Next, we demonstrate the optimality of the proposed trading strategy, taking as given the acquisition time τ. This analysis closely follows the proof in Back and Baruch (2004). Define $V(\pi) \equiv (\xi_h - \xi_l) \int_\pi^1 \frac{1-a}{\lambda(a)} da$ and consider the proposed post-acquisition value function for the case $S = h$ (the case for $S = l$ is analogous)

$$J^h (\pi_t, \Delta_t) = \Delta_t V(\pi_t).$$

We begin by showing that the given J characterizes the value function for $t \geq \tau$. Consider $\{t : \tau \leq t < T\}$ and suppose $S = h$. Direct calculation on the function V yields

$$V' = (\xi_h - \xi_l) \frac{\pi - 1}{\lambda} \tag{19}$$

$$rV = \frac{1}{2}\sigma_Z^2\lambda^2 V'', \tag{20}$$

which coincides with eq. (1.15) and (1.16) in Back and Baruch (2004).

Let θ_t denote an arbitrary admissible trading strategy. Following Back and Baruch (2004), let $\hat{\pi}_t$ denote the process defined by $\hat{\pi}_s = \hat{\alpha}$ for $s \leq \tau$ and $d\hat{\pi}_t = \lambda(\hat{\pi})dY_t$ for $t > \tau$ and $0 < \hat{\pi}_t < 1$, with Y_t generated when the trader follows the given arbitrary trading strategy. In order to condense notation, in this section, we denote $\mathbb{E}[\cdot|\mathcal{F}_t^P] = \mathbb{E}_t[\cdot]$. Since θ is admissible, we know that

$$\mathbb{E}_\tau\left[\int_\tau^T \Delta_u(1-\pi_u)\theta_u^- du\right] = \mathbb{E}_\tau\left[\int_\tau^\infty e^{-r(u-\tau)}\Delta_u(1-\hat{\pi}_u)\theta_u^- du\right] < \infty,$$

from which it follows that

$$\int_\tau^\infty e^{-r(u-\tau)}\Delta_u(1-\hat{\pi}_u)\theta_u^- du < \infty$$

almost surely, and therefore that the integral

$$\int_\tau^\infty e^{-r(u-\tau)}\Delta_u(1-\hat{\pi}_u)\theta_u du$$

is well-defined, though is possibly infinite.

Let $\hat{T} = \inf\{t \geq \tau : \hat{\pi} \in \{0,1\}\}$. Applying Ito's lemma to $e^{-r(t-\tau)}J$ yields

$$\begin{aligned}
&e^{-r(t\wedge\hat{T}-\tau)}J^h(\hat{\pi}_{t\wedge\hat{T}}, \Delta_{t\wedge\hat{T}}) - J^h(\hat{\pi}_\tau, \Delta_\tau) \\
&\quad= \int_\tau^{t\wedge\hat{T}} e^{-r(u-\tau)}\Delta\left(-rV(\hat{\pi}_u) + \lambda\theta V'(\hat{\pi}_u) + \frac{1}{2}\sigma_Z^2\lambda^2 V''(\hat{\pi}_u)\right)du \\
&\qquad\qquad + \sigma_Z\int_\tau^{t\wedge\hat{T}} e^{-r(u-\tau)}\Delta\lambda V'(\hat{\pi}_u)dW_{Zu} + \sigma_N\int_\tau^{t\wedge\hat{T}} e^{-r(u-\tau)}\Delta V(\hat{\pi}_u)dW_{\Delta u} \\
&\quad= -(\xi_h - \xi_l)\int_\tau^{t\wedge\hat{T}} e^{-r(u-\tau)}\Delta_u\theta_u(1-\hat{\pi}_u)du - \sigma_Z(\xi_h-\xi_l)\int_\tau^{t\wedge\hat{T}} e^{-r(u-\tau)}\Delta_u(1-\hat{\pi}_u)dW_{Zu} \\
&\qquad\qquad + \sigma_\Delta\int_\tau^{t\wedge\hat{T}} e^{-r(u-\tau)}\Delta_u V(\hat{\pi}_u)dW_{\Delta u}
\end{aligned} \tag{21}$$

where the last equality uses eq. (19) and (20). Since $V \geq 0$, the above implies

$$(\xi_h - \xi_l) \int_\tau^{t \wedge \hat{T}} e^{-r(u-\tau)} \Delta_u \theta_u (1 - \hat{\pi}_u) du \leq \Delta_\tau V(\alpha) + x(t), \tag{22}$$

where we define $x(t) = \sigma_\Delta \int_\tau^{t \wedge \hat{T}} e^{-r(u-\tau)} \Delta_u V(\hat{\pi}_u) dW_{\Delta u} - \sigma_Z (\xi_h - \xi_l) \int_\tau^{t \wedge \hat{T}} e^{-r(u-\tau)} \Delta_u (1 - \hat{\pi}_u) dW_{Zu}$. The integrands in the stochastic integrals are locally bounded and hence the integrals are local martingales (Thm. 29, Ch. 4, Protter (2003)). It follows that $x(t)$ is itself a local martingale (Thm. 48, Ch. 1, Protter (2003)).

Let $\hat{\tau}_n$ be a localizing sequence of stopping times for $x(t)$. That is, $\hat{\tau}_{n+1} \geq \hat{\tau}_n$, $\hat{\tau}_n \to \infty$, and $x(t \wedge \hat{\tau}_n)$ is a martingale for each n. Because $x(t)$ is a local martingale such a sequence exists (e.g., because $x(t)$ is continuous we can take $\hat{\tau}_n = \inf\{t : |x(t)| \geq n\}$). Further considering the sequence $n \wedge \hat{\tau}_n$, eq. (22) implies

$$(\xi_h - \xi_l) \int_\tau^{n \wedge \hat{\tau}_n \wedge \hat{T}} e^{-r(u-\tau)} \Delta_u \theta_u (1 - \hat{\pi}_u) du \leq \Delta_\tau V(\alpha) + x(n \wedge \hat{\tau}_n).$$

Applying Fatou's lemma,[30] along with this inequality, yields

$$\begin{aligned}
\mathbb{E}_\tau \left[(\xi_h - \xi_l) \int_\tau^{\hat{T}} e^{-r(u-\tau)} \Delta_u \theta_u (1 - \hat{\pi}_u) du \right] &\leq \liminf_{n \to \infty} \mathbb{E}_\tau \left[(\xi_h - \xi_l) \int_\tau^{n \wedge \hat{\tau}_n \wedge \hat{T}} e^{-r(u-\tau)} \Delta_u \theta_u (1 - \hat{\pi}_u) du \right] \\
&\leq \Delta_\tau V(\alpha) + \liminf_{n \to \infty} \mathbb{E}_\tau \left[x(n \wedge \hat{\tau}_n) \right] \\
&\leq \Delta_\tau V(\alpha).
\end{aligned}$$

Note that for $\hat{T} < \infty$ we have $\hat{\pi}_{\hat{T}} = 1$ since $\hat{\pi}_{\hat{T}} = 0$ would imply a violation of the admissibility condition. To establish this, note that eq. (21) implies

$$-\mathbb{E}_\tau \left[(\xi_h - \xi_l) \int_\tau^{n \wedge \hat{\tau}_n \wedge \hat{T}} e^{-r(u-\tau)} \Delta_u \theta_u (1 - \hat{\pi}_u) du \right] = \mathbb{E}_\tau \left[e^{-r(t \wedge \hat{T} - \tau)} \Delta_{t \wedge \hat{T}} V(\hat{\pi}_{t \wedge \hat{T}}) - \Delta_\tau V(\alpha) \right] - J^h(\hat{\pi}_\tau, \Delta_\tau),$$

and therefore

$$- \mathbb{E}_\tau \left[(\xi_h - \xi_l) \int_\tau^{\hat{T}} e^{-r(u-\tau)} \Delta_u \theta_u (1 - \hat{\pi}_u) du \right]$$

[30] The typical formulation of Fatou's Lemma requires that the integrands f_n be weakly positive. However, if f_n^- is bounded above by an integrable function g, considering $f_n + g$ in Fatou's lemma delivers the result. Here, due to the admissibility condition we can take $g = N_u(1 - p_u)\theta_u^-$.

$$
\begin{aligned}
&\geq \limsup_{n\to\infty} \mathbb{E}_\tau \left[-(\xi_h - \xi_l) \int_\tau^{n\wedge\hat{\tau}_n\wedge\hat{T}} e^{-r(u-\tau)} \Delta_u \theta_u (1-\hat{\pi}_u) du \right] \\
&= \limsup_{n\to\infty} \mathbb{E}_\tau \left[e^{-r(n\wedge\hat{\tau}_n\wedge\hat{T}-\tau)} \Delta_{n\wedge\hat{\tau}_n\wedge\hat{T}} V(\hat{\pi}_{n\wedge\hat{\tau}_n\wedge\hat{T}}) - \Delta_\tau V(\alpha) \right] - J^h(\hat{\pi}_\tau, \Delta_\tau) \\
&\geq \mathbb{E}_\tau \left[e^{-r(\hat{T}-\tau)} \Delta_{\hat{T}} V(\hat{\pi}_{\hat{T}}) \right] - J^h(\hat{\pi}_\tau, \Delta_\tau) \\
&= \infty,
\end{aligned}
$$

where the first line applies the 'reverse' Fatou's Lemma, the second line uses the equality in the previous displayed equation, the third line applies Fatou's Lemma and the final line follows because $V(0) = \infty$. Furthermore, $\hat{\pi}_u = \hat{\pi}_{\hat{T}} = 1$ for all $u \geq \hat{T}$ since 1 is an absorbing state. It follows that

$$
\mathbb{E}_\tau \left[(\xi_h - \xi_l) \int_\tau^\infty e^{-r(u-\tau)} \Delta_u \theta_u (1-\hat{\pi}_u) du \right] = \mathbb{E}_\tau \left[(\xi_h - \xi_l) \int_\tau^{\hat{T}} e^{-r(u-\tau)} \Delta_u \theta_u (1-\hat{\pi}_u) du \right] \leq \Delta_\tau V(\alpha). \tag{23}
$$

Furthermore, this inequality is trivially true for $\hat{T} = \infty$, so it holds regardless of the behavior of $\hat{T}$. It follows that

$$
\Delta_\tau V(\alpha) \geq \mathbb{E}_\tau \left[(\xi_h - \xi_l) \int_\tau^\infty e^{-r(u-\tau)} \Delta_u \theta_u (1-\hat{\pi}_u) du \right] = \mathbb{E}_\tau \left[(\xi_h - \xi_l) \int_\tau^T \Delta_u \theta_u (1-\pi_u) du \right],
$$

since $\hat{\pi} = \pi$ for $t \leq T$. Hence $\Delta_\tau V(\alpha)$ is an upper bound on the post-acquisition value function.

To establish the optimality of the trader's post-acquisition strategy and the expression for the value function, it remains to show that the expected profits generated by the strategy attain the bound $\Delta_\tau V(\alpha)$. (We show below that the trader's overall trading strategy is admissible.) Compute the trader's expected profit at time τ. We have

$$
\begin{aligned}
\mathbb{E}_\tau \left[(\xi_h - \xi_l) \int_\tau^T \theta^h(\pi_u) \Delta_u (1-\pi_u)\, du \right] &= \int_\tau^\infty (\xi_h - \xi_l) \mathbb{E}_\tau \left[\mathbf{1}_{\{t\leq T\}} \theta^h(\pi_u) \Delta_u (1-\pi_u) \right] du \\
&= \int_\tau^\infty (\xi_h - \xi_l) \mathbb{E}_\tau[\Delta_u] \mathbb{E}_\tau \left[\mathbf{1}_{\{t\leq T\}} \theta^h(\pi_u)(1-\pi_u) \right] du \\
&= (\xi_h - \xi_l) \Delta_\tau \int_\tau^\infty \mathbb{E}_\tau \left[\mathbf{1}_{\{t\leq T\}} \theta^h(\pi_u)(1-\pi_u) \right] du \\
&= (\xi_h - \xi_l) \Delta_\tau \mathbb{E}_\tau \left[\int_\tau^T \theta^h(\pi_u)(1-\pi_u)\, du \right],
\end{aligned}
$$

where the first equality applies Fubini's theorem which is permissible because the integrand is positive, the second equality uses the fact that N is independent of T and $\{p_u\}$, the next-to-last equality follows because N is a martingale, and the final equality applies Fubini's theorem again. The proof in Back and Baruch (2004) establishes that under the given trading strategy and pricing rule, $V(\alpha) = \mathbb{E}_\tau\left[(\xi_h - \xi_l)\int_\tau^T \theta^h(\pi_u)(1-\pi_u)\,du\right]$. Hence,

$$\Delta_\tau V(\alpha) = \mathbb{E}_\tau\left[(\xi_h - \xi_l)\int_\tau^T \theta^h(\pi_u)\Delta_u(1-\pi_u)\,du\right],$$

which establishes the optimality of the post-acquisition trading strategy.

Let $J^U(\Delta)$ denote the pre-acquisition value function (i.e., the value function for a trader prior to information acquisition and entry). Note that because $\pi \equiv \hat{\alpha}$ for $t < \tau$, J^U effectively depends only on the news process in this case. We need to characterize this function and establish that the overall posited trading strategy, involving no trade prior to acquisition, is optimal. Under the given trading strategy, we have

$$\begin{aligned} J^U(\Delta) &= \mathbb{E}\left[\mathbf{1}_{\{\tau<T\}}(\xi_h - \xi_l)\int_\tau^T \theta^S(\pi_u)\Delta_u(\mathbf{1}_{S=h} - \pi_u)\,du\right] \\ &= \mathbb{E}\left[\mathbf{1}_{\{\tau<T\}} J^S(\pi_\tau, \Delta_\tau)\right] \end{aligned}$$

Let $\check{\theta}$ be any admissible trading strategy that is adapted to $\mathcal{F}_t^P$ and $\hat{\theta}$ any admissible strategy that is adapted to $\mathcal{F}_t^I$. Then $\theta = \mathbf{1}_{\{t<\tau\}}\check{\theta} + \mathbf{1}_{\{t\geq\tau\}}\hat{\theta}$ is an arbitrary admissible strategy that obeys the restriction that the investor does not observe ξ until time τ. The expected profits from following this strategy are

$$\begin{aligned} &\mathbb{E}_0\left[\mathbf{1}_{\{\tau<T\}}\int_0^\tau \check{\theta}_u\Delta_u(\xi - \alpha)\,du + \mathbf{1}_{\{\tau<T\}}(\xi_h - \xi_l)\int_\tau^T \hat{\theta}_u\Delta_u(\mathbf{1}_{S=h} - \pi_u)\,du + \mathbf{1}_{\{\tau\geq T\}}\int_0^T \check{\theta}_u\Delta_u(\xi - \alpha)\,du\right] \\ &= \mathbb{E}_0\left[\mathbf{1}_{\{\tau<T\}}(\xi_h - \xi_l)\int_\tau^T \hat{\theta}_u\Delta_u(\xi - \pi_u)\,du\right] \\ &= \mathbb{E}_0\left[\mathbf{1}_{\{\tau<T\}}\mathbb{E}\left[(\xi_h - \xi_l)\int_\tau^T \hat{\theta}_u\Delta_u(\xi - \pi_u)\,du|\mathcal{F}_\tau^I\right]\right] \\ &\leq \mathbb{E}_0\left[\mathbf{1}_{\{\tau<T\}} J^S(\pi_\tau, \Delta_\tau)\right] \\ &= J^U(\Delta), \end{aligned}$$

where the first equality takes expectations over ξ, the second equality uses the law of iterated expectations, and the inequality follows since it was shown above that as of time τ, our posited trading strategy achieves higher expected profit than any other admissible strategy. $\square$

Proof of Proposition 2. Let $\bar{J}(\Delta_t)$ denote the value of entry at instant $t$when the news process is equal to Δ_t. Using the expression for the post-acquisition value function in Proposition 1, we have

$$\bar{J}\left(\Delta_t\right) = \Delta_t(\xi_h - \xi_l)\left(\hat{\alpha}\int_{\alpha}^{1}\frac{1-a}{\lambda(a)}\,da + (1-\hat{\alpha})\int_{0}^{\alpha}\frac{a}{\lambda(a)}\,da\right) \equiv \Delta_t K.$$

Make the change of variables $x = \Phi^{-1}(1-a)$ in the integrals in the expression for $J^U(N_t)$

$$\begin{aligned}
K &= (\xi_h - \xi_l)\left(\hat{\alpha}\sqrt{\tfrac{\sigma_Z^2}{2r}}\int_{\hat{\alpha}}^{1}(1-a)\frac{1}{\phi(\Phi^{-1}(1-a))}\,da + (1-\alpha)\sqrt{\tfrac{\sigma_Z^2}{2r}}\int_{0}^{\hat{\alpha}} a\frac{1}{\phi(\Phi^{-1}(1-a))}\,da\right)\\
&= (\xi_h - \xi_l)\left(-\hat{\alpha}\sqrt{\tfrac{\sigma_Z^2}{2r}}\int_{\Phi^{-1}(1-\alpha)}^{-\infty}\Phi(x)dx - (1-\hat{\alpha})\sqrt{\tfrac{\sigma_Z^2}{2r}}\int_{\infty}^{\Phi^{-1}(1-\alpha)}(1-\Phi(x))\,dx\right)\\
&= (\xi_h - \xi_l)\left(\hat{\alpha}\sqrt{\tfrac{\sigma_Z^2}{2r}}\int_{-\infty}^{\Phi^{-1}(1-\alpha)}\Phi(x)dx + (1-\hat{\alpha})\sqrt{\tfrac{\sigma_Z^2}{2r}}\int_{\Phi^{-1}(1-\alpha)}^{\infty}(1-\Phi(x))\,dx\right).
\end{aligned}$$

Now integrate by parts

$$\begin{aligned}
K &= (\xi_h - \xi_l)\left(\hat{\alpha}\sqrt{\tfrac{\sigma_Z^2}{2r}}\int_{-\infty}^{\Phi^{-1}(1-\hat{\alpha})}\Phi(x)dx + (1-\hat{\alpha})\sqrt{\tfrac{\sigma_Z^2}{2r}}\int_{\Phi^{-1}(1-\hat{\alpha})}^{\infty}(1-\Phi(x))\,dx\right)\\
&= (\xi_h - \xi_l)\hat{\alpha}\sqrt{\tfrac{\sigma_Z^2}{2r}}\left(-\int_{-\infty}^{\Phi^{-1}(1-\hat{\alpha})}x\phi(x)\,dx + x\Phi(x)\Big|_{-\infty}^{\Phi^{-1}(1-\hat{\alpha})}\right)\\
&\qquad + (\xi_h - \xi_l)(1-\hat{\alpha})\sqrt{\tfrac{\sigma_Z^2}{2r}}\left(\int_{\Phi^{-1}(1-\hat{\alpha})}^{\infty}x\phi(x)\,dx + x(1-\Phi(x))\Big|_{\Phi^{-1}(1-\hat{\alpha})}^{\infty}\right)\\
&= (\xi_h - \xi_l)\hat{\alpha}\sqrt{\tfrac{\sigma_Z^2}{2r}}\left(-\int_{-\infty}^{\Phi^{-1}(1-\hat{\alpha})}x\phi(x)\,dx + (1-\hat{\alpha})\Phi^{-1}(1-\hat{\alpha})\right)\\
&\qquad + (\xi_h - \xi_l)(1-\hat{\alpha})\sqrt{\tfrac{\sigma_Z^2}{2r}}\left(\int_{\Phi^{-1}(1-\hat{\alpha})}^{\infty}x\phi(x)\,dx - \hat{\alpha}\Phi(1-\hat{\alpha})\right)\\
&= (\xi_h - \xi_l)\sqrt{\tfrac{\sigma_Z^2}{2r}}\int_{-\infty}^{\Phi^{-1}(1-\hat{\alpha})}-x\phi(x)\,dx = (\xi_h - \xi_l)\sqrt{\tfrac{\sigma_Z^2}{2r}}\phi(\Phi^{-1}(1-\hat{\alpha})),
\end{aligned}$$

since $\int -x\phi(x)dx = \int \phi'(x)dx = \phi\left(x\right)$.

The pre-entry value function under optimal stopping is

$$J^U(\delta) \equiv \sup_{\tau\in\mathcal{T}}\mathbb{E}\left[\mathbf{1}_{\{\tau<T\}}(K\Delta_\tau - c) \mid \Delta_t = \delta\right] = \sup_{\tau\in\mathcal{T}}\mathbb{E}\left[e^{-r\tau}(K\Delta_\tau - c)^+ \mid \Delta_t = \delta\right],$$

where the second equality follows because T is independently exponentially distributed and it suffices to consider only the positive part of $K\Delta_\tau - c$ since the trader can always guarantee herself zero profit by not acquiring. Note that this problem is similar to pricing a perpetual American call option on an asset with price process KN_t that follows a geometric Brownian motion and with strike price c. Hence, standard results (Peskir and Shiryaev (2006), Chapter 4) imply that there is a uniquely optimal stopping time and this time is is a first hitting time of the N_t process,

$$T_\Delta = \inf\{t > 0 : \Delta_t \geq \Delta^*\},$$

where $\Delta^* > 0$ is a constant to be determined.

The value function and optimal N^* solve the following free boundary problem

$$\begin{aligned}
rJ^U &= \tfrac{1}{2}\sigma_\Delta^2 \Delta_t^2 J^U_{\Delta\Delta} && \text{for } \delta < \Delta^* \\
J^U(\Delta^*) &= K\Delta^* - c && \text{`value matching'} \\
J^U_\Delta(\Delta^*) &= K && \text{`smooth pasting'} \\
J^U(\delta) &> (\delta - c)^+ && \text{for } \delta < \Delta^* \\
J^U(\delta) &= (\delta - c)^+ && \text{for } \delta > \Delta^* \\
J^U(\delta) &= 0.
\end{aligned}$$

To determine the solution in the continuation region $\delta < \Delta^*$, consider a trial solution of the form $J^U(\delta) = A\delta^\beta$. Substituting and matching terms in the differential equation yields

$$r = \tfrac{1}{2}\sigma_\Delta^2 \beta(\beta - 1), \quad \beta = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1 + \tfrac{8r}{\sigma_\Delta^2}}$$

and the boundary condition at $\Delta = 0$ requires that one take the positive root

$$\beta = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 + \tfrac{8r}{\sigma_\Delta^2}}.$$

Applying the above conjecture to the value-matching and smooth pasting conditions implies:

$$\Delta^* = \frac{\beta}{\beta - 1}\frac{c}{K}, \qquad A = \frac{K}{\beta}\left(\frac{\beta}{\beta - 1}\frac{c}{K}\right)^{1-\beta} = \frac{c}{\beta - 1}\frac{1}{(\Delta^*)^\beta},$$

and the resulting function satisfies $J^U(\delta) > \delta - c$ in the continuation region, which establishes the result. The comparative statics with respect to c, σ_Δ, and σ_Z are immediate from the explicit expression for Δ^*.

To establish the remaining results, note that q and α appear only in K, so their effects on the boundary will follow from establishing their effects on K. Straightforward algebra shows that

$$\xi_h - \xi_l = \frac{\alpha(1-\alpha)(2q-1)}{q(1-q)+\alpha(1-\alpha)(2q-1)^2} = \frac{2q-1}{\frac{q(1-q)}{\alpha(1-\alpha)}+(2q-1)^2}$$

Therefore, we have

$$K = \frac{2q-1}{\frac{q(1-q)}{\alpha(1-\alpha)}+(2q-1)^2}\sqrt{\frac{\sigma_Z^2}{2r}}\phi\left(\Phi^{-1}\left(1-\hat{\alpha}\right)\right).$$

It is now immediate that K is hump-shaped and symmetric around 1/2 in α.[31] Therefore the optimal acquisition boundary is U-shaped in α and symmetric around $\alpha = 1/2$.

To establish the result for q, define

$$f \equiv \frac{\alpha(1-\alpha)(2q-1)}{q(1-q)+\alpha(1-\alpha)(2q-1)^2} \tag{24}$$

$$g \equiv \phi\left(\Phi^{-1}\left(1-\hat{\alpha}\right)\right) = \phi\left(\Phi^{-1}\left(\alpha(1-q)+(1-\alpha)q\right)\right) \tag{25}$$

and note that

$$K = \sqrt{\frac{\sigma_Z^2}{2r}}fg. \tag{26}$$

Taking the log of K and differentiating with respect to q, it is equivalent to sign

$$\frac{\partial \log K}{\partial q} = \frac{f_q}{f} + \frac{g_q}{g}. \tag{27}$$

It is straightforward to show that $f_q \geq 0$ and $g_q \leq 0$, so the sign depends on the relative sizes of these two terms.

We have

$$\frac{f_q}{f} = \frac{1-2q+2q^2-2\alpha(1-\alpha)(2q-1)^2}{(2q-1)\left(q(1-q)+\alpha(1-\alpha)(2q-1)^2\right)} \tag{28}$$

$$\geq \frac{1-2q+2q^2-\frac{1}{2}(2q-1)^2}{(2q-1)\left(q(1-q)+\frac{1}{4}(2q-1)^2\right)} \tag{29}$$

$$= \frac{2}{2q-1}, \tag{30}$$

[31] Recall that $\phi\left(\Phi^{-1}\left(\cdot\right)\right)$ is hump-shaped around 1/2 and also that $\phi\left(\Phi^{-1}\left(1-\hat{\alpha}\right)\right) = \phi\left(\Phi^{-1}\left(\hat{\alpha}\right)\right)$), which establishes that the $\phi(\Phi^{-1}(\cdot))$ term is hump-shaped in α since replacing $\hat{\alpha}$ with $1-\hat{\alpha}$ simply exchanges which of α and $(1-\alpha)$ multiplies q and $1-q$ in this term.

where the second line follows from taking $\alpha = 1/2$ and the final line does some tedious algebra.

Turning to g, we can bound the magnitude of the derivative term. Without loss of generality, due to the symmetry of K in α, suppose $\alpha \leq 1/2$, which implies $1 - \hat{\alpha} \geq 1/2$. We have

$$\frac{|g_q|}{g} \leq \frac{|(1-2\alpha)|\,|\Phi^{-1}(1-\hat{\alpha})|}{\phi\left(\Phi^{-1}\left(1-\hat{\alpha}\right)\right)} \tag{31}$$

$$\leq \frac{|\Phi^{-1}(1-\hat{\alpha})|}{\phi\left(\Phi^{-1}\left(1-\hat{\alpha}\right)\right)} \tag{32}$$

$$= \frac{\Phi^{-1}(1-\hat{\alpha})}{\phi\left(\Phi^{-1}\left(1-\hat{\alpha}\right)\right)}. \tag{33}$$

The upper-tail inequality for the standard normal pdf implies that for $x \geq 0$

$$1 - \Phi(x) \leq \frac{\phi(x)}{x} \Rightarrow \frac{x}{\phi(x)} \leq \frac{1}{1-\Phi(x)}. \tag{34}$$

Setting $x = \Phi^{-1}(1-\hat{\alpha}) \geq \Phi^{-1}(1/2) = 0$ in this inequality yields

$$\frac{\Phi^{-1}(1-\hat{\alpha})}{\phi(\Phi^{-1}(1-\hat{\alpha}))} \leq \frac{1}{1-\Phi(\Phi^{-1}(1-\hat{\alpha}))} \tag{35}$$

$$= \frac{1}{\hat{\alpha}} \tag{36}$$

$$\leq 2, \tag{37}$$

where the final line uses $\hat{\alpha} \leq 1/2$.

Putting things together,

$$\frac{\partial \log K}{\partial q} = \frac{f_q}{f} + \frac{g_q}{g} \tag{38}$$

$$\geq \frac{2}{2q-1} - 2 \tag{39}$$

$$\geq 0, \tag{40}$$

where the final line uses $1/2 \leq q \leq 1$. Hence, K is increasing in q and therefore Δ^* is decreasing in q.

Moreover, since

$$\frac{\partial}{\partial r}\Delta^* = \frac{c}{\sigma_Z^2 \phi\left(\Phi^{-1}\left(1-\hat{\alpha}\right)\right)} \frac{4\sqrt{2}\left(\sqrt{r} - 2\sqrt{\frac{r}{\frac{8r}{\sigma_\Delta^2}+1}}\right)}{\left(\sigma_\Delta - \sqrt{\sigma_\Delta^2 + 8r}\right)^2} \tag{41}$$

we know that Δ^* is decreasing in r when $r < \frac{3}{8}\sigma_\Delta^2$, but increasing otherwise. Because the trading horizon $h = 1/r$ is strictly decreasing and maps $(0, \infty)$ onto itself, it follows that the derivative of the acquisition boundary with respect to the horizon is also first decreasing and then increasing. This is easily established by appealing to the result for r and writing $r = 1/h$. □

Proof of Proposition 3. In what follows, it is useful to define T_Δ as the first time $\Delta_t \geq \Delta^*$. Then, the time at which information is acquired can be expressed as

$$\tau = T_\Delta 1_{\{T_\Delta \leq T\}} + \infty \times 1_{\{T_\Delta > T\}}, \tag{42}$$

where, as before, $\tau = \infty$ corresponds to no entry. To avoid the trivial case, assume $\Delta_0 \equiv 1 < \Delta^*$. We begin with the following observation.

Lemma 2.

Suppose $1 < \Delta^*$. For $0 \leq t < \infty$, the probability that $T_\Delta \in [t, t+dt]$ is given by

$$\Pr\left(T_\Delta \in [t, t+dt]\right) = \frac{\log\left(\Delta^*\right)}{\sigma_\Delta\sqrt{2\pi t^3}} \exp\left\{-\frac{\left(\frac{1}{\sigma_\Delta}\log\left(\Delta^*\right) + \frac{1}{2}\sigma_\Delta^2 t\right)^2}{2t}\right\} dt. \tag{43}$$

The probability that T_Δ is not finite is given by $\Pr\left(T_\Delta = \infty\right) = 1 - \frac{1}{\Delta^*}$.

Proof. Note that

$$\begin{aligned} \Delta_t \geq \Delta^* &\iff \log(\Delta_t) \geq \log(\Delta^*) \\ &\iff -\frac{1}{2}\sigma_\Delta t + W_{\Delta t} \geq \frac{1}{\sigma_\Delta}\log(\Delta^*), \end{aligned}$$

so that the first time that Δ_t hits Δ^* is the first time that a Brownian motion with drift $-\frac{1}{2}\sigma_\Delta$ hits $\frac{1}{\sigma_\Delta}(\log(\frac{\Delta^*}{\Delta_0}))$. It follows from Karatzas and Shreve (1998) (Chapter 3.5, Part C, p.196-197) that for $\Delta_0 < \Delta^*$ the density of T_Δ is

$$\Pr\left(T_\Delta \in [t, t+dt]\right) = \frac{\left(\log\left(\Delta^*\right)\right)}{\sigma_\Delta \sqrt{2\pi t^3}} \exp\left\{ -\frac{\left(\frac{1}{\sigma_\Delta} \log\left(\Delta^*\right) + \frac{1}{2}\sigma_\Delta t\right)^2}{2t} \right\} dt.$$

Moreover, since $\frac{1}{\sigma_\Delta}\log(\Delta^*) > 0$ but the drift is $-\frac{1}{2}\sigma_\Delta < 0$, it follows from Karatzas and Shreve (1998) (p.197) that $\Pr(T_\Delta = \infty) > 0$. Specifically, note that

$$\Pr\left(T_\Delta < \infty\right) = \int_0^\infty \frac{\log\left(\Delta^*\right)}{\sigma_\Delta \sqrt{2\pi t^3}} \exp\left\{ -\frac{\left(\frac{1}{\sigma_\Delta} \log\left(\Delta^*\right) + \frac{1}{2}\sigma_\Delta t\right)^2}{2t} \right\} dt = \frac{1}{\Delta^*}, \tag{44}$$

which implies $\Pr\left(T_\Delta = \infty\right) = 1 - \frac{1}{\Delta^*}$. □

Given the definition of τ, we have that for $0 \leq t < \infty$,

$$\begin{aligned}
\Pr\left(\tau \in [t, t+dt]\right) = & \begin{array}{c} \Pr\left(\tau \in [t, t+dt] \,\middle|\, T_\Delta \leq T\right) \Pr\left(T_\Delta \leq T\right) \\ + \Pr\left(\tau \in [t, t+dt] \,\middle|\, T_\Delta > T\right) \Pr\left(T_\Delta > T\right) \end{array} && (45) \\
&= \Pr\left(T_\Delta \in [t, t+dt] \,\middle|\, T_\Delta \leq T\right) \Pr\left(T_\Delta \leq T\right) && (46) \\
&= \Pr\left(T_\Delta \in [t, t+dt]\right) \Pr\left(T \geq t\right) && (47) \\
&= e^{-rt} \Pr\left(T_\Delta \in [t, t+dt]\right). && (48)
\end{aligned}$$

Integrating gives us

$$\begin{aligned}
\Pr\left(\tau < \infty\right) &= \int_0^\infty e^{-rt} \frac{\log(\Delta^*)}{\sigma_\Delta \sqrt{2\pi t^3}} \exp\left\{ -\frac{\left(\frac{1}{\sigma_N} \log\left(\Delta^*\right) + \frac{1}{2}\sigma_\Delta t\right)^2}{2t} \right\} dt && (49) \\
&= \frac{e^{-\frac{\log(\Delta^*)\sqrt{8r+\sigma_\Delta^2}}{2\sigma_\Delta}}}{\sqrt{\Delta^*}} = \left(\frac{1}{\Delta^*}\right)^\beta && (50)
\end{aligned}$$

The comparative statics for c, σ_Z, q, and α follow from plugging in the expressions for Δ^* and β, and using the results from Proposition 2 for the effect of q and α on Δ^*. To establish the comparative statics for σ_Δ, first note that since $\lim_{\sigma_\Delta \to 0} \beta = \infty$, $\lim_{\sigma_\Delta \to \infty} \beta = 1$, and $\Delta^* = \frac{\beta}{\beta - 1} \frac{c}{K}$,

$$\lim_{\sigma_\Delta \to \infty} \Pr\left(\tau < \infty\right) = 0 \tag{51}$$

$$\lim_{\sigma_\Delta \to 0} \Pr(\tau < \infty) = \begin{cases} 0 & \text{if } c > K \\ 1 & \text{if } c \le K \end{cases}. \tag{52}$$

Let

$$\zeta \equiv \tfrac{\partial}{\partial \beta}\left(\log\left(\Pr(\tau < \infty)\right)\right) = \log\left(\tfrac{1}{\Delta^*}\right) + \beta \tfrac{\partial}{\partial \beta} \log\left(\tfrac{1}{\Delta^*}\right) = \log\left(\tfrac{1}{\Delta^*}\right) + \tfrac{1}{\beta - 1} \tag{53}$$

which implies $\lim_{\sigma_\Delta \to 0} \zeta = \lim_{\beta \to \infty} \zeta = \log\left(\frac{K}{c}\right)$, $\lim_{\sigma_\Delta \to \infty} \zeta = \lim_{\beta \to 1} \zeta = \infty$, and

$$\tfrac{\partial}{\partial \sigma_\Delta} \zeta = \tfrac{\partial \zeta}{\partial \beta} \tfrac{\partial \beta}{\partial \sigma_\Delta} = -\tfrac{1}{\beta(1-\beta)^2} \tfrac{\partial \beta}{\partial \sigma_\Delta} > 0. \tag{54}$$

Since $\frac{\partial}{\partial \sigma_\Delta} \log\left(\Pr(\tau < \infty)\right) = \zeta \frac{\partial \beta}{\partial \sigma_\Delta}$, we have the following results:

- When $c \le K$, since $\zeta \ge 0$ for $\sigma_\Delta \to 0$ and $\frac{\partial}{\partial \sigma_\Delta}\zeta > 0$ we have $\zeta > 0$ for all σ_Δ, which in turn implies $\frac{\partial}{\partial \sigma_\Delta} \log\left(\Pr(\tau < \infty)\right) < 0$ for all σ_Δ.
- When $c > K$, ζ crosses zero once, from below, as σ_Δ increases, which implies $\frac{\partial}{\partial \sigma_\Delta} \log\left(\Pr(\tau < \infty)\right) = 0$ at exactly this one point. In this case, $\Pr(\tau < \infty)$ is hump-shaped.

Similarly, for r, $\frac{\partial}{\partial r} \log(\Pr(\tau < \infty)) = \zeta \frac{\partial}{\partial r}\beta - \frac{\beta}{2r}$. We have $\frac{\partial}{\partial r}\zeta = -\frac{1}{\beta(\beta-1)^2} \frac{\partial}{\partial r}\beta - \frac{1}{2r} < 0$. Since $\frac{\partial}{\partial r}\beta = \frac{1}{\sigma_\Delta^2\left(\beta - \frac{1}{2}\right)} > 0$ this implies $\frac{\partial}{\partial r} \log(\Pr(\tau < \infty))$ crosses zero as most once as r increases and from above if it does so. Consider the limit as r tends to zero,

$$\begin{aligned} \lim_{r \to 0} \tfrac{\partial}{\partial r} \log(\Pr(\tau < \infty)) &= \lim_{r \to 0} \left(\zeta \tfrac{\partial}{\partial r}\beta - \tfrac{\beta}{2r}\right) \\ &= \lim_{r \to 0} \frac{2r\zeta - \sigma_\Delta^2 \beta \left(\beta - \frac{1}{2}\right)}{2\sigma_\Delta^2 r \left(\beta - \frac{1}{2}\right)}. \end{aligned} \tag{55}$$

If it can be shown that the numerator in eq. (55) has a finite, positive limit it will follow that the overall limit is ∞. Considering the numerator, we have

$$\begin{aligned} \lim_{r \to 0} \left(2r\zeta - \sigma_\Delta^2 \beta \left(\beta - \tfrac{1}{2}\right)\right) &= 2 \lim_{r \to 0} r \left(\tfrac{1}{\beta - 1} - \log \tfrac{\beta}{\beta - 1} - \log \sqrt{2r}\right) - \tfrac{1}{2}\sigma_\Delta^2 \\ &= \sigma_\Delta^2 - 2 \lim_{r \to 0} \frac{\frac{1}{\beta(\beta-1)}}{\frac{1}{r^2}} - \tfrac{1}{2}\sigma_\Delta^2 \\ &= \tfrac{1}{2}\sigma_\Delta^2 - 2 \lim_{r \to 0} \frac{2r}{(2\beta - 1)\frac{\partial}{\partial r}\beta} = \tfrac{1}{2}\sigma_\Delta^2 \end{aligned}$$

where the second equality applies l'Hôspital's rule to the three different terms and uses the fact $\frac{\partial}{\partial r}\beta \to \frac{2}{\sigma_\Delta^2}$ as $\beta \to 1$. The third equality rearranges the expression in the remaining limit to place r^2 in the numerator and uses l'Hôspital's rule again. Returning to eq. (55), this implies $\lim_{r \to 0} \frac{\partial}{\partial r} \log(\mathbb{P}(\tau < \infty)) = \infty$.

Now, consider $\lim_{r\to\infty} \frac{\partial}{\partial r} \log(\Pr(\tau < \infty))$. We have

$$\lim_{r\to\infty} \zeta = \lim_{r\to\infty} \left(\tfrac{1}{\beta-1} - \log \tfrac{\beta}{\beta-1} - \log \sqrt{2r} \right) = \lim_{\beta\to\infty} \left(\tfrac{1}{\beta-1} - \log \tfrac{\beta}{\beta-1} \right) - \lim_{r\to\infty} \log \sqrt{2r} = -\infty.$$

Because $\frac{\partial}{\partial r}\beta > 0$, it follows that $\lim_{r\to\infty} \frac{\partial}{\partial r} \log(\Pr(\tau < \infty)) = -\infty$. Because the trading horizon $h = 1/r$ is strictly decreasing and maps $(0, \infty)$ onto itself, it follows that the derivative of the acquisition probability with respect to the horizon is also first decreasing and then increasing. This is easily established by appealing to the result for r and writing $r = 1/h$. This completes the proof. $\square$

Proof of Proposition 4. It was shown in Proposition 3 that τ is infinite with strictly positive probability. Hence, all unconditional moments (of positive order) are infinite.

Using the density from eq. (49) to compute the conditional density of τ gives

$$\mathbb{P}(\tau \in [t, t + dt] | \tau < \infty) = \frac{e^{-rt} \frac{\log(\Delta^*)}{\sigma_\Delta \sqrt{2\pi t^3}} \exp\left\{ -\frac{\left(\frac{1}{\sigma_N} \log(\Delta^*) + \frac{1}{2}\sigma_\Delta t\right)^2}{2t} \right\} \mathbf{1}_{\{t<\infty\}}}{\mathbb{P}(\tau < \infty)} dt$$

Integrating and doing some simplifying algebra gives

$$\begin{aligned} \mathbb{E}[\tau | \tau < \infty] &= \int_0^\infty t \frac{e^{-rt} \frac{\log(\Delta^*)}{\sigma_\Delta \sqrt{2\pi t^3}} \exp\left\{ -\frac{\left(\frac{1}{\sigma_N} \log(\Delta^*) + \frac{1}{2}\sigma_\Delta t\right)^2}{2t} \right\}}{\mathbb{P}(\tau < \infty)} dt \\ &= \frac{2\log(\Delta^*)}{\sigma_\Delta^2 \sqrt{1 + \frac{8r}{\sigma_\Delta^2}}} \end{aligned}$$

The conditional expectation inherits the comparative statics of Δ^* with respect to c, σ_Z^2, q, and α since these parameters appear only in Δ^* and log is an increasing transformation.

To determine the dependence on σ_Δ^2, note that by solving $\beta = \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{8r}{\sigma_\Delta^2}}$ for σ_Δ^2 and substituting, we can write

$$\mathbb{E}[\tau | \tau < \infty] = 2 \frac{1}{\left(\frac{2r}{\beta(\beta-1)} \left(\frac{2r}{\beta(\beta-1)} + 8r \right) \right)^{1/2}} \log(\Delta^*) \tag{56}$$

$$= 2 \sqrt{\frac{\beta(\beta - 1)}{2r \left(\frac{2r}{\beta(\beta-1)} + 8r \right)}} \log(\Delta^*) \tag{57}$$

$$= 2 \sqrt{\frac{\beta^2(\beta - 1)^2}{4r^2 + 16r^2\beta(\beta - 1)}} \log(\Delta^*) \tag{58}$$

$$= \frac{1}{r}\sqrt{\frac{\beta^2(\beta-1)^2}{1+4\beta(\beta-1)}}\log(\Delta^*) \tag{59}$$

$$= \frac{\beta(\beta-1)}{r(2\beta-1)}\log\left(\frac{\beta}{\beta-1}\frac{c}{K}\right), \tag{60}$$

We have

$$\frac{\partial}{\partial\beta}\mathbb{E}[\tau|\tau<\infty] = \frac{1-2\beta+2\beta^2}{r(2\beta-1)^2}\log\left(\frac{\beta}{\beta-1}\frac{c}{K}\right) - \frac{\beta(\beta-1)}{r(2\beta-1)}\frac{1}{\beta(\beta-1)} \tag{61}$$

$$= \frac{(1-2\beta+2\beta^2)\log\left(\frac{\beta}{\beta-1}\frac{c}{K}\right)-(2\beta-1)}{r(2\beta-1)^2} \tag{62}$$

$$= \frac{(2\beta(\beta-1)+1)\log\left(\frac{\beta}{\beta-1}\frac{c}{K}\right)-(2\beta-1)}{r(2\beta-1)^2} \tag{63}$$

$$\tag{64}$$

The denominator of this expression is strictly positive, so the sign of the derivative depends on the sign of the numerator

$$\frac{\partial}{\partial\beta}\mathbb{E}[\tau|\tau<\infty] \gtrless 0 \iff (2\beta(\beta-1)+1)\log\left(\frac{\beta}{\beta-1}\frac{c}{K}\right) - (2\beta-1) \gtrless 0 \tag{65}$$

$$\iff \log\left(\frac{\beta}{\beta-1}\frac{c}{K}\right) \gtrless \frac{(2\beta-1)}{(2\beta(\beta-1)+1)} \tag{66}$$

$$\iff \log\left(\frac{c}{K}\right) \gtrless \underbrace{\frac{(2\beta-1)}{(2\beta(\beta-1)+1)} - \log\left(\frac{\beta}{\beta-1}\right)}_{\equiv g(\beta)} \tag{67}$$

Calculating the derivative of g, we have

$$g'(\beta) = \frac{(2\beta-1)^2}{\beta(\beta-1)(2\beta(\beta-1)+1)^2} > 0. \tag{68}$$

Furthermore,

$$\lim_{\beta\downarrow 1} g(\beta) = -\infty \tag{69}$$

$$\lim_{\beta\to\infty} g(\beta) = 0. \tag{70}$$

Hence, $g(\beta) < 0$ for any $\beta > 1$ and strictly increases towards 0 as β increases.

Since β is strictly decreasing in σ_Δ^2, this implies

$$\frac{d}{d\sigma_\Delta^2} g(\beta) = \frac{\partial g}{\partial \beta} \underbrace{\frac{\partial \beta}{\partial \sigma_\Delta^2}}_{<0} < 0 \tag{71}$$

$$g(\beta) \mid_{\sigma_\Delta^2 \to \infty} = -\infty \tag{72}$$

$$g(\beta) \mid_{\sigma_\Delta^2 \to 0} = 0. \tag{73}$$

Putting things together, if $c \geq K$ then it is immediate that

$$\log\left(\frac{c}{K}\right) \geq 0 > g(\beta), \tag{74}$$

so that $\mathbb{E}[\tau|\tau < \infty]$ is decreasing in σ_Δ^2.

On the other hand, if $c < K$, then $\log\left(\frac{c}{K}\right) < 0$ and therefore for sufficiently small σ_Δ^2 we have

$$0 \gtrapprox g(\beta) > \log\left(\frac{c}{K}\right), \tag{75}$$

so that $\mathbb{E}[\tau|\tau < \infty]$ is increasing in σ_Δ^2 for small . For sufficiently large σ_Δ^2 we have

$$g(\beta) < \log\left(\frac{c}{K}\right) \tag{76}$$

so that $\mathbb{E}[\tau|\tau < \infty]$ is decreasing in σ_Δ^2. Given the monotonicity of $g(\beta)$ there is a unique β at which the sign of the dependence flips and therefore we conclude that if $c \geq K$ then $\mathbb{E}[\tau|\tau < \infty]$ is decreasing in σ_Δ^2, and if $c < K$ then $\mathbb{E}[\tau|\tau < \infty]$ is first increasing in σ_Δ^2 and then decreasing.

To determine the dependence on r, let $\kappa = (\xi_h - \xi_l)\sqrt{\sigma_Z^2}\phi\left(\Phi^{-1}(1-\hat{\alpha})\right)$, which does not depend on r, and recall that

$$\mathbb{E}[\tau|\tau < \infty] = \frac{\beta(\beta-1)}{r(2\beta-1)} \log\left(\frac{\beta}{\beta-1}\frac{c}{K}\right) = \frac{1}{r}\frac{\beta(\beta-1)}{(2\beta-1)} \log\left(\frac{\beta}{\beta-1}\frac{c\sqrt{2r}}{\kappa}\right) \tag{77}$$

Taking the total derivative with respect to r gives

$$\frac{d}{dr}\mathbb{E}[\tau|\tau < \infty] = \frac{\partial \mathbb{E}[\tau|\tau < \infty]}{\partial r} + \frac{\partial \mathbb{E}[\tau|\tau < \infty]}{\partial \beta}\frac{\partial \beta}{\partial r} \tag{78}$$

$$= -\frac{1}{r^2}\frac{\beta(\beta-1)}{(2\beta-1)} \log\left(\frac{\beta}{\beta-1}\frac{c\sqrt{2r}}{\kappa}\right) + \frac{1}{r}\frac{\beta(\beta-1)}{(2\beta-1)}\frac{1}{2r} \tag{79}$$

$$+\frac{\partial}{\partial\beta}\left(\frac{\beta(\beta-1)}{r(2\beta-1)}\log\left(\frac{\beta}{\beta-1}\frac{c}{K}\right)\right)\frac{\partial\beta}{\partial r} \tag{80}$$

$$=-\frac{1}{r^2}\frac{\beta(\beta-1)}{(2\beta-1)}\log\left(\frac{\beta}{\beta-1}\frac{c}{K}\right)+\frac{1}{r}\frac{\beta(\beta-1)}{(2\beta-1)}\frac{1}{2r} \tag{81}$$

$$+\frac{\frac{2\beta(\beta-1)+1}{2\beta-1}\log\left(\frac{\beta}{\beta-1}\frac{c}{K}\right)-1}{r(2\beta-1)}\frac{\partial\beta}{\partial r} \tag{82}$$

We have

$$\frac{\partial\beta}{\partial r}=\frac{2}{\sigma_\Delta^2\sqrt{1+\frac{8r}{\sigma_\Delta^2}}} \tag{83}$$

$$=\frac{2}{\sigma_\Delta^2}\frac{1}{2\beta-1} \tag{84}$$

$$=\frac{\beta(\beta-1)}{r(2\beta-1)}, \tag{85}$$

where the second equality substitutes for σ_Δ^2 in terms of β.

Plugging in to the previous expression gives

$$\frac{d}{dr}\mathbb{E}[\tau|\tau<\infty]=-\frac{1}{r^2}\frac{\beta(\beta-1)}{(2\beta-1)}\log\left(\frac{\beta}{\beta-1}\frac{c}{K}\right)+\frac{1}{r}\frac{\beta(\beta-1)}{(2\beta-1)}\frac{1}{2r} \tag{86}$$

$$+\frac{\frac{2\beta(\beta-1)+1}{2\beta-1}\log\left(\frac{\beta}{\beta-1}\frac{c}{K}\right)-1}{r(2\beta-1)}\frac{\beta(\beta-1)}{r(2\beta-1)} \tag{87}$$

$$=\frac{\beta(\beta-1)}{r^2(2\beta-1)^2}\left(\frac{-2\beta(\beta-1)}{(2\beta-1)}\log\left(\frac{\beta}{\beta-1}\frac{c}{K}\right)+\frac{2\beta-3}{2}\right) \tag{88}$$

Hence,

$$\frac{d}{dr}\mathbb{E}[\tau|\tau<\infty]\gtrless 0 \iff \frac{-2\beta(\beta-1)}{(2\beta-1)}\log\left(\frac{\beta}{\beta-1}\frac{c}{K}\right)+\frac{2\beta-3}{2}\gtrless 0 \tag{89}$$

$$\iff \frac{(2\beta-3)(2\beta-1)}{4\beta(\beta-1)}-\log\left(\frac{\beta}{\beta-1}\right)-\log\left(\frac{c}{\kappa}\sqrt{2r}\right)\gtrless 0 \tag{90}$$

Rearranging the expression for β yields

$$\sqrt{2r}=\frac{1}{2}\sqrt{\sigma_\Delta^2}\sqrt{(2\beta-1)^2-1} \tag{91}$$

Hence,

$$\frac{d}{dr}\mathbb{E}[\tau|\tau<\infty] \gtrless 0 \iff \underbrace{\frac{(2\beta-3)(2\beta-1)}{4\beta(\beta-1)} - \log\left(\frac{\beta}{\beta-1}\right) - \log\left(\frac{c}{2\kappa}\sqrt{\sigma_\Delta^2}\sqrt{(2\beta-1)^2-1}\right)}_{\equiv h(\beta)} \gtrless 0. \tag{92}$$

Hence, the desired sign depends on the sign of $h(\beta)$ for $\beta \in (1,\infty)$. To begin, we will find its maximum. Differentiate h and set equal to zero

$$h'(\beta) = \frac{3-6\beta+4\beta^2}{4\beta^2(\beta-1)^2} + \frac{1}{\beta(\beta-1)} - \frac{2\beta-1}{2\beta(\beta-1)} = 0 \tag{93}$$

$$\iff 3-6\beta+4\beta^2+4\beta(\beta-1)-2\beta(\beta-1)(2\beta-1) = 0 \tag{94}$$

$$\iff -(2\beta-1)(2\beta^2-6\beta+3) = 0 \tag{95}$$

This is a cubic with one (real) root at $\beta = 1/2$, and two others (both real), which the quadratic formula gives as

$$\beta = \frac{1}{2}(3 \pm \sqrt{3}). \tag{96}$$

The root $\beta = \frac{1}{2}(3+\sqrt{3})$ is the only one in the interval $\beta \in (1,\infty)$, so it is the only relevant critical point. The second-order condition is

$$\frac{3-12\beta+18\beta^2-12\beta^3+2\beta^4}{2\beta^3(\beta-1)^3} \leq 0, \tag{97}$$

and at $\beta = \frac{1}{2}(3+\sqrt{3})$ we have

$$\left.\frac{3-12\beta+18\beta^2-12\beta^3+2\beta^4}{2\beta^3(\beta-1)^3}\right|_{\beta=\frac{1}{2}(3+\sqrt{3})} = 2-\frac{4}{\sqrt{3}} < 2-\frac{4}{\sqrt{4}} = 0. \tag{98}$$

Hence, h has a local maximum at $\beta = \frac{1}{2}(3+\sqrt{3})$. At the boundaries, we have

$$\lim_{\beta\downarrow 1} h(\beta) = -\infty \tag{99}$$

$$\lim_{\beta\to\infty} h(\beta) = -\infty. \tag{100}$$

Because h is negatively infinite at the boundaries and has a single critical point in $(1,\infty)$, it follows that it achieves a *global* maximum at $\beta = \frac{1}{2}(3+\sqrt{3})$. The value of h at this point is

$$h\left(\frac{1}{2}(3+\sqrt{3})\right) = \frac{1}{2}\underbrace{\left(1+\log\left(\frac{1}{18}(2\sqrt{3}-3)\right)\right)}_{<0} - \log\left(\frac{c}{2\kappa\sqrt{\sigma_\Delta^2}}\right). \tag{101}$$

If $\frac{c}{2\kappa\sqrt{\sigma_\Delta^2}}$ is sufficiently large, this expression is negative, $h < 0$ for all $\beta \in (1,\infty)$, and therefore for all $r > 0$. Hence, in this case $\frac{d}{dr}\mathbb{E}[\tau|\tau < \infty] < 0$. On the other hand, if $\frac{c}{2\kappa\sqrt{\sigma_\Delta^2}}$ is not sufficiently large then h is first negative, then positive, then negative, as β increases, which implies that $\frac{d}{dr}\mathbb{E}[\tau|\tau < \infty]$ changes sign from negative to positive and back to negative as r increases. This completes the proof. $\square$

Proof of Proposition 5 When evaluating entry when $\Delta_t = \Delta$, the payoff can be written concisely as

$$\max\{K_h\Delta - c_h, K_l\Delta - c_l, 0\}$$

Since the payoff function is monotonic in Δ, standard results imply that the optimal entry time is a first hitting time, from below, for the Δ process. The value function J^U and entry boundary Δ^* satisfy

$$\begin{aligned}
rJ^U &= \tfrac{1}{2}\sigma_\Delta^2\Delta^2 J_{\Delta\Delta} && \text{continuation region}\\
J^U(\Delta^*) &= \max\{K_h\Delta^* - c_h, K_l\Delta^* - c_l, 0\}\\
J^U_\Delta(\Delta^*) &= \left.\frac{d}{d\delta}\right|_{\delta=\Delta^*} \max\{K_h\delta - c_h, K_l\delta - c_l, 0\}\\
J^U(\delta) &> \max\{K_h\delta - c_h, K_l\delta - c_l, 0\} && \text{continuation region}\\
J^U(\delta) &= \max\{K_h\delta - c_h, K_l\delta - c_l, 0\} && \text{outside continuation region}\\
J^U(0) &= 0
\end{aligned}$$

Consider a trial solution of the form $J^U(\delta) = A\delta^\beta$. Note that for there to be a single optimal acquisition point Δ^* it must either acquire a low precision signal in the region $K_h\Delta - c_H \le K_l\Delta - c_l \iff \Delta \le \frac{c_h - c_l}{K_h - K_l} \equiv \bar{\Delta}$ or a high precision signal in the region $\Delta \ge \bar{\Delta}$.

Let's first search for a low-precision solution. Plugging the conjectured solution into the differential equation yields

$$\beta = \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{8r}{\sigma_\Delta^2}}.$$

The value-matching and smooth-pasting conditions require

$$\begin{aligned} A_l &= \frac{K_l}{\beta}\left(\frac{c_l}{K_l}\frac{\beta}{\beta-1}\right)^{1-\beta} \\ \Delta_l^* &= \frac{c_l}{K_l}\frac{\beta}{\beta-1}. \end{aligned}$$

Now, let's search for a high-precision solution. The differential equation still implies

$$\beta = \frac{1}{2} + \frac{1}{2}\sqrt{1+\frac{8r}{\sigma_\Delta^2}}.$$

The value-matching and smooth-pasting conditions imply

$$\begin{aligned} A_h &= \frac{K_h}{\beta}\left(\frac{c_h}{K_h}\frac{\beta}{\beta-1}\right)^{1-\beta} \\ \Delta_h^* &= \frac{c_h}{K_h}\frac{\beta}{\beta-1}. \end{aligned}$$

The choice between a low- and high-precision signal therefore depends on which of the value functions from the two candidate solutions is larger. Comparing the value functions from implies that the choice of precision depends on

$$K_h^\beta c_h^{1-\beta} \gtreqless K_l^\beta c_l^{1-\beta},$$

as well as whether the candidate Δ_j^* is greater or less than $\bar{\Delta} = \frac{c_h - c_l}{K_h - K_l}$ as discussed above.

Note that a high-precision signal is always optimal if $c_h/K_h < c_l/K_l$ since in that case we have

$$\begin{aligned} c_h/K_h < c_l/K_l &\Rightarrow K_l^\beta c_l^{-\beta} < K_h^\beta c_h^{-\beta} \\ &\Rightarrow K_l^\beta c_l^{1-\beta} < K_h^\beta c_l c_h^{-\beta} \\ &\Rightarrow K_l^\beta c_l^{1-\beta} < K_h^\beta c_h^{1-\beta}, \end{aligned}$$

where the second line multiplies through by $c_l > 0$ and the final line uses $c_l \leq c_h$. Furthermore, in this case

$$\bar{\Delta} = \frac{c_h - c_l}{K_h - K_l} = \frac{\frac{1}{K_l}\frac{c_h}{K_h} - \frac{1}{K_h}\frac{c_l}{K_l}}{\frac{1}{K_l} - \frac{1}{K_h}}$$

$$
\begin{aligned}
&\le \frac{\frac{1}{K_l}\frac{c_h}{K_h} - \frac{1}{K_h}\frac{c_h}{K_h}}{\frac{1}{K_l} - \frac{1}{K_h}} \\
&= \frac{c_h}{K_h} \\
&\le \frac{\beta}{\beta - 1}\frac{c_h}{K_h} = \Delta_h^*.
\end{aligned}
$$

Now, consider the case $c_h/K_h \geq c_l/K_l$. Note that in this case, the only situation in which the optimal signal is not immediate is when $\Delta_l^* \leq \bar{\Delta} \leq \Delta_h^*$. In that case, to determine the optimal signal, one must directly compare the two value functions, which reduces to comparing A_h and A_l, as well as $\Delta_l^* \leq \bar{\Delta}$, as in the Proposition. □