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Disagreement and Learning: Dynamic Patterns of Trade

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ABSTRACT

The empirical evidence on investor disagreement and trading volume is difficult to reconcile in standard rational expectations models. We develop a dynamic model in which investors disagree about the interpretation of public information. We obtain a closed-form linear equilibrium that allows us to study which restrictions on the disagreement process yield empirically observed volume and return dynamics. We show that when investors have infrequent but major disagreements, there is positive autocorrelation in volume and positive correlation between volume and volatility. We also derive novel empirical predictions that relate the degree and frequency of disagreement to volume and volatility dynamics.

THE EMPIRICAL LITERATURE on trading volume has documented a number of regularities that cannot be easily explained by standard rational expectations (RE hereafter) models in which investors share common priors and interpret information in the same way. A summary of earlier results can be found in Karpoff (1987) and Gallant, Rossi, and Tauchen (1992). More recently, Kandel and Pearson (1995) document significant abnormal trading volume around earnings announcements, even when the announcement returns are close to zero. They also find that analyst forecasts often diverge or flip around earnings announcements, which they argue is inconsistent with models in which analysts agree on the interpretation of public information. Chae (2005) further documents that abnormal volume before an earnings announcement is low, but spikes on the announcement date and decreases slowly over the next few days. While noisy RE models can generate similar patterns using specific stochastic endowment or noise trading processes, such explanations are completely driven by these exogenous and unobservable processes and hence do not provide many insights. This has led many to view trading volume to be the key ingredient missing from our theoretical models. For example, in a recent talk Cochrane (2007) suggested that the "Next Revolution" in asset pricing will consist of models that can explain empirically observed levels and patterns of trading volume.

In this paper, we take a step in this direction and develop a dynamic model of trade. Building on the differences of opinion (DO hereafter) literature, we

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consider a setup where agents disagree about the interpretation of public information. In contrast to RE models in which investors share common priors and disagree due to asymmetric information, investors in DO models have heterogeneous priors and interpret information differently. Hence, investors may "agree to disagree" even if they have the same information.[1] Our goal is to provide a simple and intuitive characterization of the volume process in such a model. We show that since investors' relative trading positions reflect the extent to which they disagree, trading volume largely reflects revisions to the level of disagreement. We also show that the equilibrium price corresponds to the average valuation across investors. Based on these results, we develop several implications that relate patterns in trading volume and return volatility to investor disagreement. This is useful in generating additional empirical predictions that potentially distinguish our model from other DO and RE models. While some of the model's predictions are consistent with existing empirical evidence, other predictions that relate the dynamics of volume and volatility to the level and frequency of disagreement are unique to this model.

In particular, we show that when investors have large but infrequent disagreements, volume exhibits positive autocorrelation and is clustered around these large disagreements. When the degree of disagreement is time varying, return volatility and volume are also positively correlated over time. These relationships among disagreement, volume, and return volatility seem natural: investors agree on the interpretation of information most of the time, and periods of high disagreement are often associated with high volume and high volatility.

Next, we extend the analysis to an infinite horizon model in order to analytically derive sharper and cleaner empirical predictions. Again, investors disagree about the interpretation of public signals, but now we can allow for periodic jumps in disagreement. We show that a large jump in volume is associated with high return volatility, high volume autocorrelation, and high expected returns. We also show that when return volatility is high, abnormal volume and volume autocorrelation are positively related, but when return volatility is low, they are negatively related. Moreover, volume autocorrelation is nonmonotonic in the frequency of jumps: autocorrelation is low when jumps in disagreement are very frequent or when they are very rare, but is higher otherwise. Finally, if investors bear aggregate risk by holding the asset, we show that expected returns are increasing in the average level of disagreement, the size of jumps in disagreement, and the frequency of these jumps. The preliminary empirical analysis that we perform suggests that the evidence is consistent with these predictions.

An implication of the fact that trade represents changes in the level of disagreement is that volume consists of two pieces: a convergence term and an idiosyncratic term. When agents agree on the interpretation of the current public signal but disagree on the interpretation of prior public information,

[1]These include Harrison and Kreps (1978), Harris and Raviv (1993), Kandel and Pearson (1995), Scheinkman and Xiong (2003), Basak (2004), and Cao and Ou-Yang (2009).

Bayesian updating leads their beliefs to converge; we call the corresponding volume "belief-convergence" trade. In contrast, when agents agree on the prior information but disagree on the interpretation of the current signal, the associated volume is called "idiosyncratic" trade. Note that in RE models, since investors have common priors and agree on the interpretation of the public signals, there is no trade of either type.

The positive autocorrelation in volume is due to belief-convergence trade. Large disagreement in the current period leads to idiosyncratic trade in the current period and belief-convergence trade in future periods. Moreover, if a period of large disagreement is followed by periods of low disagreement, future belief-convergence trades are relatively more important than future idiosyncratic trades. Volume spikes up when disagreement is large, but investors' beliefs converge and volume falls gradually over the next few periods. As a result, volume clusters around periods of high disagreement, and exhibits positive autocorrelation. Furthermore, when investors have more extreme interpretations (and so disagree more), price reactions to public signals are likely to be larger. Hence, periods of major disagreement are periods of higher volume and also of higher absolute price changes. This leads to positive time-series correlation between volatility and volume.

Standard RE models cannot generate these patterns easily. First, RE models are unable to generate public disagreement among investors. Even in noisy RE models, if investor disagreement is made public, beliefs would converge immediately and there would be no trade. This implies that RE models cannot reconcile the empirical evidence that analyst earnings forecasts, despite being public, exhibit significant dispersion, and that this dispersion is related to trading volume and return dynamics. Second, trading volume is difficult to generate in RE models. The "No-Trade Theorem" and its variants (e.g., Milgrom and Stokey (1982)) rule out trade when investors share common priors, even in the presence of asymmetric information. Noisy RE equilibrium models overcome this result by introducing noise traders or aggregate liquidity shocks.[2] However, as He and Wang (1995) show, public information leads to trade in RE models only in the presence of private information, and usually leads to a convergence of beliefs. Moreover, in contrast to what is observed empirically, they show that trade gradually increases before a public announcement, peaks at the announcement date, and then remains low thereafter. Finally, as Kandel and Pearson (1995) argue, it is difficult to generate large amounts of information-based trading without accompanying price changes in RE models.

Of course, as mentioned before, the RE framework is flexible with respect to the aggregate noise process. For example, one can generate serial correlation in volume by assuming serial correlation in the aggregate supply shocks, or can generate trade without price changes by forcing aggregate supply shocks to perfectly offset aggregate information shocks. However, this is unappealing

[2] Models that generate volume using asymmetric information and aggregate noise shocks include Grossman and Stiglitz (1980), Pfleiderer (1984), Kyle (1985), Admati and Pfleiderer (1988), Brown and Jennings (1989), Grundy and McNichols (1989), Wang (1994), and He and Wang (1995).

in terms of providing insight into what generates these patterns, since the noise process is assumed to be unexplained and exogenous. In contrast to such RE models, we remain agnostic about the disagreement process. The volume dynamics in our model follow from the Bayesian learning process that investors use in updating their beliefs.

The rest of the paper is organized as follows. Section I surveys some of the related literature. Section II describes the basic framework for the finite horizon model, discusses the assumptions, and characterizes the equilibrium. Section III derives the expression for volume, and analyzes both the autocorrelation in volume and the relationship between volume and returns in the finite horizon model. Section IV presents the results for the infinite horizon model and derives the empirical predictions of the model. Section V concludes. Unless noted otherwise, proofs are in the Appendix at the end of the article.

I. Related Literature

A number of papers study volume dynamics in heterogeneous information settings. For instance, He and Wang (1995) develop a dynamic model of trading volume with private and public information that leads to interesting patterns in trading volume. Kim and Verrecchia (1991) show that in a setup with heterogeneous private information, volume is proportional to the absolute price change and to the prior dispersion in precision. Kim and Verrecchia (1994) present a setup in which informed investors receive private signals at the same time the public signal is announced (which they interpret as information processing), and show that this can lead to higher disagreement in announcement periods. While these models all have interesting predictions about returns and volume around public announcements, they are unable to generate a number of empirically observed patterns in volume dynamics. In particular, in these models there is no trading volume due to a public announcement unless investors also have private information, and there is no trade without an associated change in price.

Morris (1995) presents an excellent overview on the role and limitations of the common prior assumption in economics, and makes a strong case for models in which agents have heterogeneous priors and differences of opinion. Moreover, as Brav and Heaton (2002) point out, models in which investors exhibit "rational structural uncertainty" and differences of opinion are often observationally and mathematically equivalent to models in which investors exhibit behavioral biases. This may increase the appeal of DO models since their predictions are robust to alternative interpretations of the underlying assumptions about unobservable investor behavior.[3] With a few notable exceptions (e.g., Harris and Raviv (1993), Morris (1994), Kandel and Pearson (1995)), however, the DO literature has focused primarily on pricing implications of heterogeneous priors (e.g., Harrison and Kreps (1978), Scheinkman and Xiong (2003), and Basak (2004)). Varian (1989) studies the role of differences of opinion on prices and volume in a static model, and shows that higher disagreement leads to higher

[3]We thank the referee for suggesting this argument.

volume. Harris and Raviv (1993) is one of the earlier papers to study the effect of differences of opinion on volume, but they assume investors are risk neutral. This leads to a binary, or "all or nothing," trading pattern in which optimistic investors hold all of the asset and pessimistic investors hold none. Moreover, trade only occurs when agents' beliefs flip—more specifically, agents trade exactly when their beliefs about the value of the asset cross each other and hence the agents agree. In contrast, our model generates trade when there is change in the level of disagreement.

Like ours, other papers have explored the effect of risk aversion on trading volume in DO models (e.g., Mayshar (1983) and Kandel and Pearson (1995)). Kandel and Pearson (1995) empirically document the relationship between volume, disagreement, and return volatility around public announcements. Among others, Bamber, Barron, and Stober (1997) extend this empirical analysis by decomposing trading volume around earnings announcements into components that are explained by dispersion in prior beliefs, changes in dispersion, and belief jumbling, even after controlling for the announcement-period price change. As Kandel and Pearson (1995) suggest, this evidence is inconsistent with standard models of rational expectations. Instead, they propose a model in which investors disagree on the interpretation of public signals, which leads in turn to trade. However, since investors in their model are myopic, they cannot study the dynamics of returns and volume around announcements. For instance, as we show in the Internet Appendix, when investors are myopic, there is no serial correlation in volume.[4]

In a recent paper, Cao and Ou-Yang (2009) also examine trading in a DO model. Apart from technical differences in modeling, the main difference lies in the different goals of the two papers. The focus of their paper is on trade across asset classes (equities and options), while our focus is on patterns in volume for a single asset. Moreover, while they allow for disagreement about the precision of public signals, this forces them to assume that investor disagreement about the mean of the public signal is deterministic. A result of this assumption is that trading volume in their model is linear in absolute contemporaneous price changes. In contrast, the relationship between volume and prices in our model is more subtle since the first is driven by differences in interpretation while the second is driven by the average interpretation. In particular, this implies that our model allows for trade even in the absence of price changes—an effect that has been empirically documented (e.g., Kandel and Pearson (1995)).

In Banerjee, Kaniel, and Kremer (2009), we develop an alternative DO model in which investors disagree on the fundamentals but learn about the beliefs of others by conditioning on prices. We show that unlike RE models, DO models with disagreement about higher-order beliefs can generate predictability in prices. While appropriate to study the effects of higher-order beliefs on the predictability of returns, the model in that paper is not well suited to study volume and disagreement dynamics. In particular, the model is restricted to three periods and disagreement about fundamentals does not change over time but rather is driven by private information. As a result, unlike the current

[4]The Internet Appendix is available at http://www.afajof.org/supplements.asp.

model, the model in Banerjee, Kaniel, and Kremer (2009) is unable to generate interesting volume or volatility dynamics or to link such dynamics to the disagreement process. We view the models in these two papers as complementary approaches to understanding the effect of differences in opinions on different aspects of financial markets.

II. Finite Horizon Model

We examine a finite horizon model with final period T. There are two investors (or two types of investors with equal population weights) indexed by $i \in \{1, 2\}$. Agents maximize CARA utility over final-period payoff, where we set agents' risk aversion to one for notational simplicity:

$$u(W_T) = -e^{-W_T}. \tag{1}$$

Agents trade two assets: a risky asset, whose final payoff D is normally distributed, and a risk-free asset, which pays one unit at time T with certainty. The risky asset is assumed to be in zero net supply. This assumption simplifies the analysis. If we were to instead assume constant aggregate supply of the risky asset, this would decrease the price by a deterministic risk premium term but would not affect the dynamics of volume. Also, volume dynamics in our model are not driven by aggregate noisy supply shocks, as they are in noisy RE models.[5] Instead, dynamic trading patterns in our model follow from the evolution of beliefs and disagreement.

Before observing any signals, the investors have prior beliefs about the final payoff D of the risky asset. These priors are given by

$$D = F + d, \quad \text{where} \quad d \sim N(0, \delta) \quad \text{and} \quad F \sim N(v_{i,0}, \rho_0), \tag{2}$$

where F is the component of the final payoff about which investors obtain signals, while d is the residual uncertainty that is not resolved until the last date.[6] In Section IV, we develop an infinite horizon version of the model where investors receive dividends in every period (similar to D) and learn about the mean dividend (i.e., F) over time.

For simplicity, we assume that investors have homogeneous, and correct, beliefs about the residual payoff d.[7] Moreover, to highlight the effect of differences in interpretation of public signals, we assume that the investors share a common prior expectation given by $v_{i,0} = v_{j,0} = 0$.[8]

[5]For instance, by assuming persistence in the aggregate supply shocks, one can generate serial correlation in volume and correlation between volume and absolute price changes in an RE model.

[6]He and Wang (1995) use a similar payoff structure in their RE model, and show that without residual uncertainty (i.e., $\delta = 0$), there is no trade unless investors receive private information.

[7]Since investors do not receive any information about d, their beliefs about it do not change over time and hence do not affect the dynamic of volume significantly.

[8]In the proofs, we also allow for investors to have different prior expectations, where the difference is normally distributed:

$$v_{i,0} - v_{j,0} \sim N(0, \sigma_0).$$

At each date $0 < t < T$, agents observe a public signal s_t and may disagree about its interpretation. In particular, investor i believes that s_t is given by

$$s_t = F + \varepsilon_t \quad \text{where} \quad \varepsilon_t \sim N(e_{i,t}, q_t), \tag{3}$$

where $e_{i,t}$ denotes investor i's interpretation of the public signal at date t. If agent i has a higher $e_{i,t}$, then he has a more negative view of the same signal. We assume that the $e_{i,t}$'s are normally distributed with zero mean and are independent of any other random variables:

$$e_{i,t} \sim N(0, \lambda_t). \tag{4}$$

As a result, there is uncertainty about what the interpretation of future signals will be. At time t, each agent observes both $e_{i,t}$'s and so there is no asymmetry of information.

The above specification implies that each investor believes that the other investor is wrong and so ignores the other investor's interpretation. This assumption is made primarily for tractability, but also allows us to develop the intuition for a setup with pure differences of opinion. In the real world, investors are likely to agree on certain things and disagree about others. Standard asymmetric information RE models focus on aspects of the world that investors agree about and thus learn about from each other. Our goal (like other models of differences of opinion) is to highlight aspects of the world that investors still disagree about after they have learned all they can from each other.[9] Moreover, while these types of beliefs can be motivated by behavioral biases or bounded rationality, they need not be. As Morris (1995) and others point out, relaxing the common prior assumption does not imply or require irrationality. The fact that sophisticated rational investors (and economists) often publicly disagree is evidence of this.

While we allow investors to disagree on the mean of the public signal, we assume that they agree on the precision of the public signal, although this is allowed to vary over time. A natural question to ask is whether this restriction can be relaxed. We are unable to allow for both heterogeneous precision and stochastic disagreement about the mean of the public signal in a tractable manner. One possible way to allow for heterogeneous precision while keeping the model solvable would be to make the interpretations $e_{i,t}$ of each investor deterministic.[10] Volume and return dynamics would then be driven by these exogenous, deterministic specifications.

This allows us to consider both the case in which investors begin with heterogeneous priors about valuations (for nonzero σ_0), and the case in which investors have common priors (when $\sigma_0 = 0$). However, the results of our model are driven by differences in interpretation of the public signals and not by differences in prior beliefs. We therefore focus on the simpler case in the body of the paper.

[9] We would like to thank the referee for this characterization.

[10] This would make the average valuation a weighted average of the valuations of each investor, where the weights would depend on the precision and the risk aversion of each investor (e.g., Kim and Verrecchia (1991)).

The variability in the distribution of ε_t over time captures the notion that all the public signals need not be from the same source. For example, the signal in a given period, s_t, might be an earnings announcement, while the signal in the next period, s_{t+1}, might be an analyst report, and so on. As mentioned above, the key assumption is that agents are allowed to have different interpretations of the same signal. While an earnings announcement of 10 cents is good news for one agent, it might be bad news for another. Agents are also allowed to have different interpretations across signals—a given agent might react positively to the earnings announcement, but negatively to the analyst report. Hence, we allow for the flexibility to model signals from different sources, with different precision, without further complicating the notation.

The generality of the model allows us to consider a wide range of disagreement patterns. Since agents disagree on the meaning of public information, it is not necessary that beliefs converge in the long run. To keep the model tractable and the intuition clear, we do not complicate investors' learning problem by explicitly allowing them to change their interpretation after learning from the others' interpretation or from past signals. However, since the model allows for time variation in investor beliefs about the precision of the public signals (i.e., q_t), it could implicitly capture such a phenomenon. By allowing for time variation in λ_t, we can model periods of uncertainty and large disagreement (high λ_t) and periods of similar interpretation and learning (low λ_t).

As a result of the different interpretations, investors hold different posterior beliefs about the distribution of F. In particular, we denote investor i's conditional beliefs at date t about F as

$$v_{i,t} = E_{i,t}[F] \quad \text{and} \quad \rho_t = \mathrm{var}_t[F]. \tag{5}$$

Since their information sets are symmetric, agents disagree regarding the mean of F but agree that the variance is given by ρ_t. Finally, we use the notation $\bar{X}_t = \frac{1}{2}(X_{1,t} + X_{2,t})$ to denote the average across investors of any random variable X_i, and the notation $\Delta X_{i,t} = X_{i,t} - \bar{X}_t$ to denote the deviation of investor i in variable X from the mean.

A. The Two-Period Case

We show that the equilibrium of the model has a simple, recursive form. In particular, we show that prices in each period are given by the investors' average valuation (i.e., $P_t = \bar{v}_t$), and the optimal demand of each investor is driven by his valuation $v_{i,t}$. To clarify the intuition for the model before presenting the main result, we explicitly derive the equilibrium for the special case where $T = 2$. We solve the model using backward induction. At date 1, the optimal demand and price are given by

$$x_{i,1} = \frac{v_{i,1} - P_1}{\rho_1 + \delta} \quad \text{and} \quad P_1 = \bar{v}_1.$$

This follows immediately from the assumptions of exponential utility and normally distributed payoffs. The price reflects the average valuation, and the optimal demand of each agent reflects the difference between his valuation and the average valuation. Based on this, we conclude that at date 0, investor i's optimal demand solves the following problem:

$$\begin{aligned} x_{i,0} &= \arg\max_x E_{i,0}[-\exp\{-x(P_1 - P_0) - x_{i,1}(F + d - P_1)\}] \\ &= \arg\max_x E_{i,0}[-\exp\{-x(P_1 - P_0)\}E_{i,1}[\exp\{-x_{i,1}(F + d - P_1)\}]] \\ &= \arg\max_x E_{i,0}\left[-\exp\left\{-x(P_1 - P_0) - \frac{1}{2(\rho_1 + \delta)}(v_{i,1} - P_1)^2\right\}\right]. \end{aligned}$$

The second equality follows from the law of iterated expectations, and the third equality follows from substituting the optimal demand at date 1 and taking expectations with respect to the investor's information set at date 1. As a result, the expected utility at date 0 depends on the price gain $P_1 - P_0$ and the conditional expected utility at date 1. At date 0, the investor forms beliefs about the value F and next period's price using Bayes's Rule. We denote these beliefs by

$$\begin{pmatrix} P_1 \\ v_{i,1} \end{pmatrix} \sim N\left(\begin{pmatrix} E_{i,0}[P_1] \\ v_{i,0} \end{pmatrix}, \begin{pmatrix} \eta_0 & \pi\rho_0 \\ \pi\rho_0 & \pi\rho_0 \end{pmatrix}\right), \quad \text{where} \quad \pi = \frac{1/q_1}{1/\rho_0 + 1/q_1}.$$

As a result, the solution to the above problem can be shown to have the following form:

$$\begin{aligned} x_{i,0} &= \omega\frac{E_{i,0}[P_1 - P_0]}{\eta_0} + (1 - \omega)\frac{v_{i,0} - P_0}{\pi\rho_0}, \quad \text{where} \\ \omega &= \frac{(\rho_1 + \delta)\eta_0}{(\rho_1 + \delta)\eta_0 + \pi\rho_0(\eta_0 - \pi\rho_0)}. \end{aligned}$$

The optimal demand is a weighted average of two components: a speculative component given by $\frac{E_{i,0}[P_1 - P_0]}{\eta_0}$ that depends on the investor's beliefs about next period's price, and a fundamental component given by $\frac{v_{i,0} - P_0}{\pi\rho_0}$ that depends on the investor's beliefs about the final payoff F. Since we know that the price at date 1 is the average valuation (i.e., $P_1 = \bar{v}_1$), investor i's conditional expectation of the price is given by

$$E_{i,0}[P_1] = (1 - \pi)\bar{v}_0 + \pi v_{i,0}.$$

In particular, note that a disagreement about the valuation between investors translates into a disagreement about next period's price. Substituting these beliefs into the optimal demand and aggregating across all investors, the market clearing condition implies that the price at date 0 is the average valuation of the asset, that is, $P_0 = \bar{v}_0$. Moreover, this implies that the speculative component of demand $E_{i,0}[P_1 - P_0]$ is a multiple of the fundamental component (note that $E_{i,0}[P_1 - P_0] = \pi(v_{i,0} - P_0)$) and so investor i's optimal demand is driven by his beliefs about the fundamental value $v_{i,0}$:

$$x_{i,0} = \phi_0(v_{i,0} - P_0).$$

We show that this generalizes to the case of $T > 2$ in the following subsection.

B. The General Case

As in the two-period case, each investor uses Bayes's Rule to update his beliefs about F using his own interpretation of the public signal. In particular, investor i's beliefs about F at date $t+1$ are given by

$$F \sim N(v_{i,t+1}, \rho_{t+1}), \tag{6}$$

where

$$v^i_{t+1} = (1-\pi_t)v_t + \pi_t(s_{t+1} - e_{i,t+1}) \quad \text{and} \quad \rho_{t+1} = \rho_t(1-\pi_t), \quad \text{with}$$
$$\pi_t = \frac{1/q_{t+1}}{1/\rho_t + 1/q_{t+1}}. \tag{7}$$

Given these beliefs, we show in the Appendix that we can characterize the equilibrium as follows:

LEMMA 1: *For all t and all investors i, in equilibrium:*

1. *Prices reflect average beliefs, that is,* $P_t = \bar{v}_t$ *for all* t,
2. *The optimal demand of investor i is of the form* $x_{it} = \phi_t(v_{i,t} - P_t) = \phi_t \Delta v_{i,t}$, *and*
3. *The expected utility of investor i is of the form* $EU_{i,t} \propto \exp\{-\frac{1}{2K_t}(v_{i,t} - P_t)^2\}$,

where, K_t *and* ϕ_t *are recursively defined in the Appendix.*

In each period, investors solve a multiple-period dynamic optimization problem. The optimal demand at date t depends not only on the investor's beliefs about the final payoff, but also on his beliefs about the price gains at each intermediate period. Given our assumptions about the conditional independence of shocks to information and interpretations, this demand takes a very simple functional form. In fact, as in the two-period example from the last section, the optimal demand at date t can be expressed as a weighted average of two components:

$$x_{i,t} = \omega_t \frac{E_{i,t}[P_{t+1} - P_t]}{\eta_t} + (1-\omega_t)\frac{E_{i,t}[F - P_t]}{\pi_t \rho_t}, \quad \text{where}$$
$$\omega_t = \frac{K_{t+1}\eta_t}{K_{t+1}\eta_t + (\pi_t\rho_t)(\eta_t - \pi_t\rho_t)}. \tag{8}$$

The first part, $\frac{E_{i,t}[P_{t+1}-P_t]}{\eta_t}$, is the speculative component of demand since it depends on beliefs about next period's price P_{t+1}. The second part, $\frac{E_{i,t}[F-P_t]}{\pi_t\rho_t}$, represents a fundamental motive for trade as it depends on beliefs about the final payoff. Since the price gain $P_{t+1} - P_t$ can be expressed as

$$P_{t+1} - P_t = (1-\pi_t)\bar{v}_t + \pi_t(s_{t+1} - \bar{e}_{t+1}) - P_t \tag{9}$$

$$= (1-\pi_t)\bar{v}_t + \pi_t(F + \varepsilon_{t+1} - \bar{e}_{t+1}) - P_t, \tag{10}$$

and since future interpretations ($e_{i,t+1}$'s) and signal noise (ε_{t+1}'s) are independent of current information, beliefs about $P_{t+1} - P_t$ are a linear function of $E_{i,t}[F - P_t]$. In particular, we show in the Appendix that

$$E_{i,t}[P_{t+1} - P_t] = \pi_t E_{i,t}[F - P_t], \tag{11}$$

and consequently the optimal demand at date t is linear in $v_{i,t} - P_t$. Market clearing implies that the date t price is given by the average valuation ($\bar{v}_t$), and substituting these into the objective function gives us the quadratic form for the objective function. Finally, we show in the Appendix that if there is no residual uncertainty (i.e., $\delta = 0$), the speculative component of trade is zero and investors trade as if they are myopic.

COROLLARY 1: *If there is no residual uncertainty (i.e., $\delta = 0$) or there is no disagreement (i.e., $\lambda_t = 0$ for all t), then the optimal demand is the myopic one, that is,*

$$x_{i,t} = \frac{1}{\rho_t}(v_{i,t} - P_t). \tag{12}$$

In either of these cases, the optimal demand in each period reduces to that of myopic investors. As we show in the Appendix, in these cases there is also no serial correlation in volume.

III. Volume

Our main focus in this paper is on volume and its properties. We define the signed trade of investor i between dates t and $t+1$ as the change in the investor's position in the risky asset during that period, that is, $x_{i,t+1} - x_{i,t}$. Given our characterization of the equilibrium in Lemma 1, we know that the price at date t is the average valuation of investors (i.e., $P_t = \bar{v}_t$) and investor i's optimal demand is given by

$$x_{i,t} = \phi_t(v_{i,t} - P_t) = \phi_t \Delta v_{i,t}. \tag{13}$$

In particular, investor i's holdings depend on the difference between his valuation and the other investors' valuation. This is intuitive, since if investor 1 is more optimistic than investor 2 (i.e., $\Delta v_{1,t} > 0$), then investor 1 is long in the risky asset while investor 2 is short in the risky asset. As a result, investor i's signed trade depends on the difference in current valuations and the difference in future interpretations:

$$x_{i,t+1} - x_{i,t} = \phi_{t+1}\Delta v_{i,t+1} - \phi_t \Delta v_{i,t}. \tag{14}$$

Since the volume in this economy is given by the absolute value of the signed trade, we have the following result.

PROPOSITION 1: *The volume at time $t+1$ is linear in the difference in prior beliefs and the difference in interpretation of new information, and is given by the expression*

$$\text{Vol}_{t+1} \equiv |x_{i,t+1} - x_{i,t}| = \left| \underbrace{(\phi_{t+1}(1-\pi_t) - \phi_t)\Delta v_{i,t}}_{\text{belief convergence term}} - \underbrace{\phi_{t+1}\pi_t \Delta e_{i,t+1}}_{\text{idiosyncratic term}} \right|. \quad (15)$$

Moreover, if $\lambda_t = \lambda$ *and* $q_t = q$, *then* $\phi_{t+1}(1-\pi_t) - \phi_t \leq 0$.

Volume is driven by two factors: the difference in prior beliefs about the value Δv_t and the difference in the shocks to interpretation Δe_{t+1}. The first term is what we refer to as the belief-convergence, or learning, term, while the second piece is referred to as the idiosyncratic term. The intuition behind these two terms is as follows. Suppose there is little difference in beliefs before the current period t, that is, the Δv's and Δe's have been small. Further, suppose there is a large shock to the differences in interpretation (high Δe_t). Investors update their beliefs using Bayes's Rule and this leads to a large difference in beliefs today (high Δv_t). The volume between periods $t-1$ and t is being driven primarily by the idiosyncratic term $\phi_t \pi_{t-1} \Delta e_t$. In the next period, $t+1$, suppose further that the shock to interpretations is small (small Δe_{t+1}). Agents interpret the public signal similarly, and the Bayesian updating leads to a convergence of beliefs. In some sense, there is little uncertainty about the interpretation of the public signal, and both agents learn about the final value of the asset.[11] This learning leads to a convergence in positions and the resulting volume between periods t to $t+1$ is driven by $(\phi_{t+1}(1-\pi_t) - \phi_t)\Delta v_t$. Hence, we call it the learning, or belief-convergence, term.

More directly, consider the following. In the event that $\Delta e_{t+1} = 0$, agents interpret the new information identically. The agents learn the same thing about the final payoff, and this leads to a convergence in beliefs about the final payoff. As their beliefs get closer to each other, the agents decrease their prior holdings (in absolute value). While hard to prove analytically, one can verify numerically that $\phi_{t+1}(1-\pi_t) - \phi_t \leq 0$ under quite general conditions. This change in positions leads to volume over time. As we discuss in the next subsection, this is also the source of the autocorrelation in volume. In contrast, if $\Delta v_t = 0$ (e.g., there have been no disagreements in the past), then the only source of volume is Δe_{t+1}. Agents will change their positions if they interpret the new information differently and hence update about the final payoff differently. As expected, $-\phi_{t+1}\pi_t \leq 0$, since a higher e_{t+1} implies a more pessimistic interpretation. The larger the difference in interpretations, the larger the consequent difference in valuation and the larger the positions taken by the agents. Finally, since the e_{t+1}'s are independent over time, the idiosyncratic term cannot be directly responsible for the autocorrelation in volume.

[11]When agents have similar interpretations about the public signal, and updating leads to a convergence in beliefs, we say that the agents "learn" about the final value of the asset. However, it might be the case that they are equally wrong in the interpretation of the signal, and therefore learn incorrectly.

A. Autocorrelation in Volume

Turning attention to the autocorrelation in volume, recall that volume is given by the expression

$$\mathrm{Vol}_{t+1} = |(\phi_{t+1}(1-\pi_t) - \phi_t)\Delta v_{i,t} - \phi_{t+1}\pi_t \Delta e_{i,t+1}|, \tag{16}$$

where $(\phi_{t+1}(1-\pi_t) - \phi_t) \leq 0$. As discussed above, the only source of autocorrelation in volume is the $(\phi_{t+1}(1-\pi_t) - \phi_t)\Delta v_t$ terms, since the Δe_{t+1} terms are serially independent. For high positive autocorrelation, one would need the Δv_t terms to be positively autocorrelated, and large in comparison to the Δe_{t+1} terms. Intuitively, high volume today is indicative of a large difference of interpretation today. If the shocks to the difference in interpretation (Δe) are small in the future, this implies more convergence, and consequently more belief-convergence trade, in the future.

Note that the $(\phi_{t+1}(1-\pi_t) - \phi_t)\Delta v_t$ terms are autocorrelated since the Δv_t terms are autocorrelated. If there is a large initial realization of Δe_t followed by a series of small realizations of Δe_{t+1}, then we should observe positive autocorrelation in volume. We can generate the above pattern with occasional large jumps in λ_t (high realizations of Δe_{t+1}) followed by long periods of low λ_t (low realizations of Δe_{t+1}). When agents disagree occasionally but agree most of the time, volume exhibits positive autocorrelation.

PROPOSITION 2: *Expected volume is given by*

$$E[\mathrm{Vol}_{t+1}] = E[|x_{i,t+1} - x_{i,t}|] = \sqrt{\frac{2}{\pi}\mathrm{var}[x_{i,t+1} - x_{i,t}]} \tag{17}$$

and the serial correlation in volume is given by

$$\mathrm{corr}[\mathrm{Vol}_{t+2}, \mathrm{Vol}_{t+1}] = \Psi(\mathrm{cov}(x_{i,t+2} - x_{i,t+1}, x_{i,t+1} - x_{i,t})), \tag{18}$$

where $\Psi(\cdot)$ is a function, symmetric around zero, defined in the Appendix.

In closed form, the expression for autocorrelation is difficult to analyze. Hence, we numerically examine the effect of the parameters on expected volume and volume autocorrelation and present the results through a set of graphs. Specifically, we examine two different effects: (1) raising a parameter in the first period while keeping it fixed in subsequent periods, and (2) raising the level of the parameter in all periods. In the case of the variance of the signal (q), the overall level has a larger impact on the autocorrelation of volume. In the case of the dispersion of beliefs (λ), however, temporary shocks play a more important role. Hence, a pattern of large occasional disagreements followed by periods of learning leads to higher volume autocorrelation. We present the results in Figures 1 and 2. The base values for the model parameters in the numerical exercise are as follows: $T = 12$, $\rho_0 = 1$, $q_t = 0.1$, and $\lambda_t = 1$. We use a log-scale for the x-axis to show the effect of large variation in the dependent variable. When looking at our results one should keep in mind that daily autocorrelation in volume is estimated to be around 0.3 to 0.4 (e.g., Llorente et al.

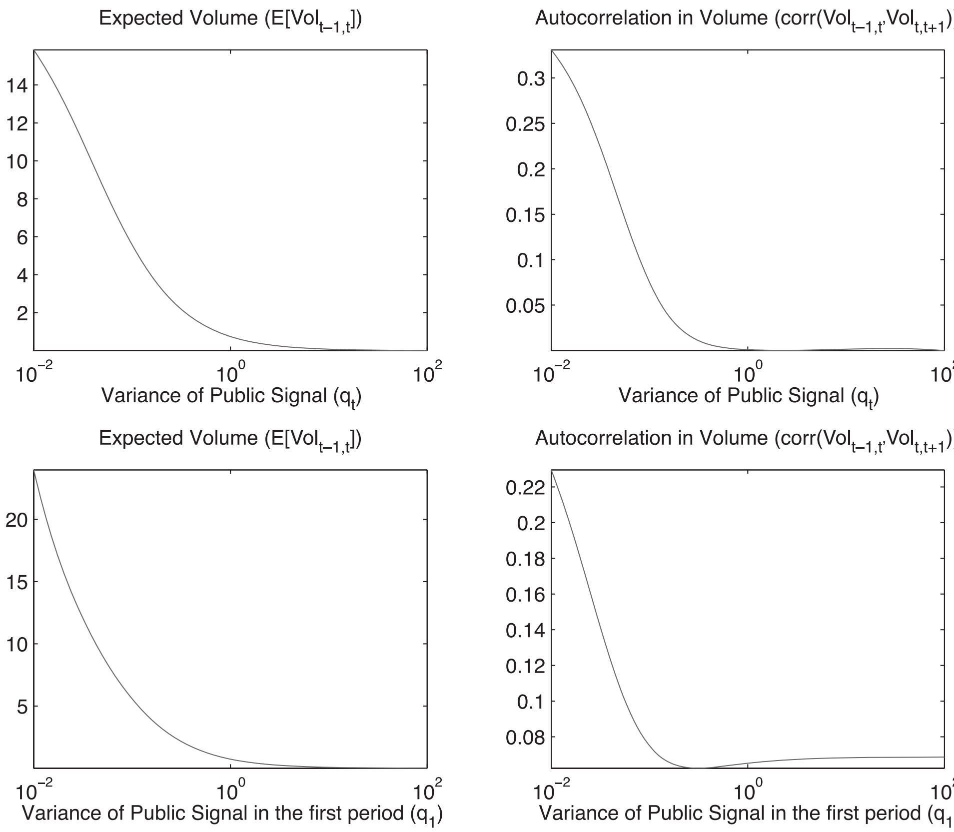

Figure 1. This figure gives expected volume and serial correlation in volume as a function of changes in the overall level of q_t and jumps in q_1 only. Other model parameters are set to the following: $\rho_0 = 1,\ q_t = 0.1,\ \lambda_t = 1,\ \delta = 1$. Note that the x-axis is in log scale.

(2002)). Results regarding the level of volume are harder to compare since in our model there is zero net supply.

For the variance of the public signal, q, the two effects are in the same direction. The top panel of Figure 1 shows that increasing the overall variance marginally decreases the expected volume and the volume autocorrelation. A higher overall variance for the public signals leads the agents to put less weight on them while updating, and this leads to smaller changes in beliefs and lower expected volume. The bottom panel shows that raising the variance of the first signal to q_1 lowers volume and autocorrelation. A high value of q_1 means that the public signal in the first period is noisier. The public signal in the second period is relatively less noisy, and updating on this new information leads to volume and price changes between the two periods.

Figure 2 shows the effects of an overall change in the degree of disagreement versus a temporary shock to disagreement. The top panel shows that increasing the variance of the differences of opinion (increasing λ_t) leads to higher

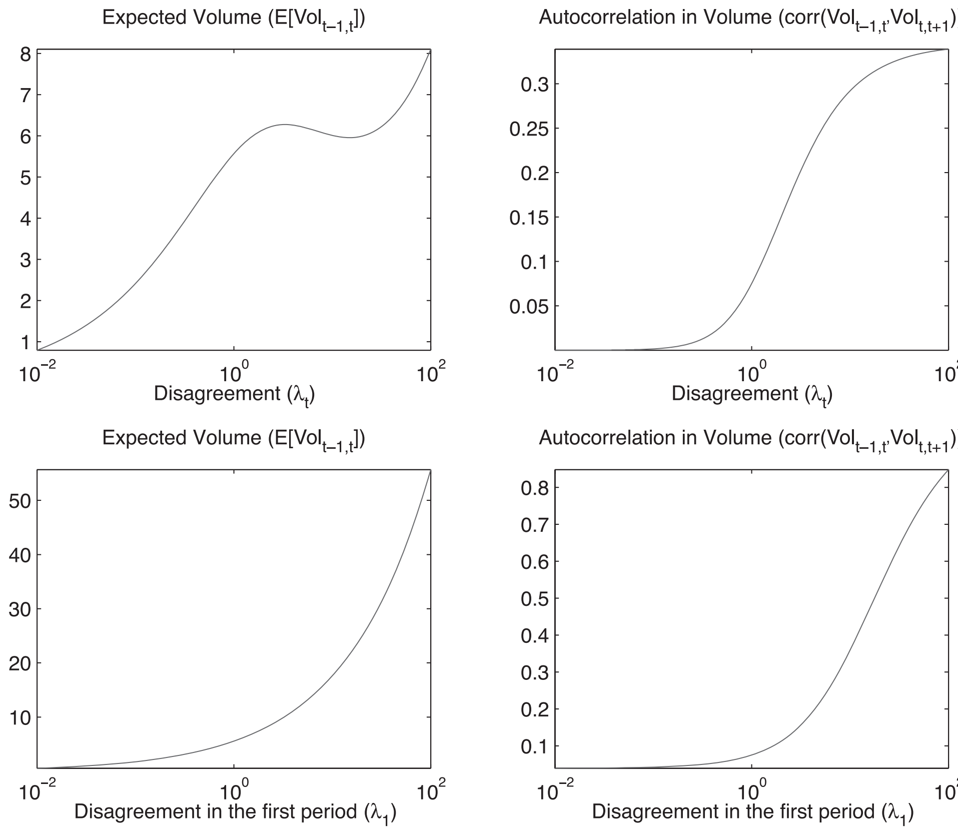

Figure 2. This figure gives expected volume and serial correlation in volume as a function of changes in the overall level of λ_t and jumps in period λ_1 only. Other model parameters are set to the following: $\rho_0 = 1,\ q_t = 0.1$, and $\lambda_t = 1,\ \delta = 1$. Note that the x-axis is in log scale.

autocorrelation in volume. However, expected volume is nonmonotonic in λ_t. This is because higher levels of overall disagreement (higher λ_t) also imply higher differences in future interpretation, which leads to higher uncertainty and less aggressive trading. As a result, expected volume is increasing in λ_t for low and high values of λ_t, but is decreasing for some intermediate levels of λ_t. Again, changing the overall level of λ_t has a small effect on autocorrelation while raising the overall level of disagreement contributes to the expected volume to some extent.

The bottom panel of Figure 2 captures the effect of large initial disagreements followed by periods of relative agreement and learning. As we suggested earlier, large initial disagreement about the interpretation of the signal (due to high λ_1) leads to a divergence of opinions. This leads agents to hold more extreme positions. In the following periods, agents have more similar interpretations (relatively low λ_t), and hence they learn from the public signals. This leads to a

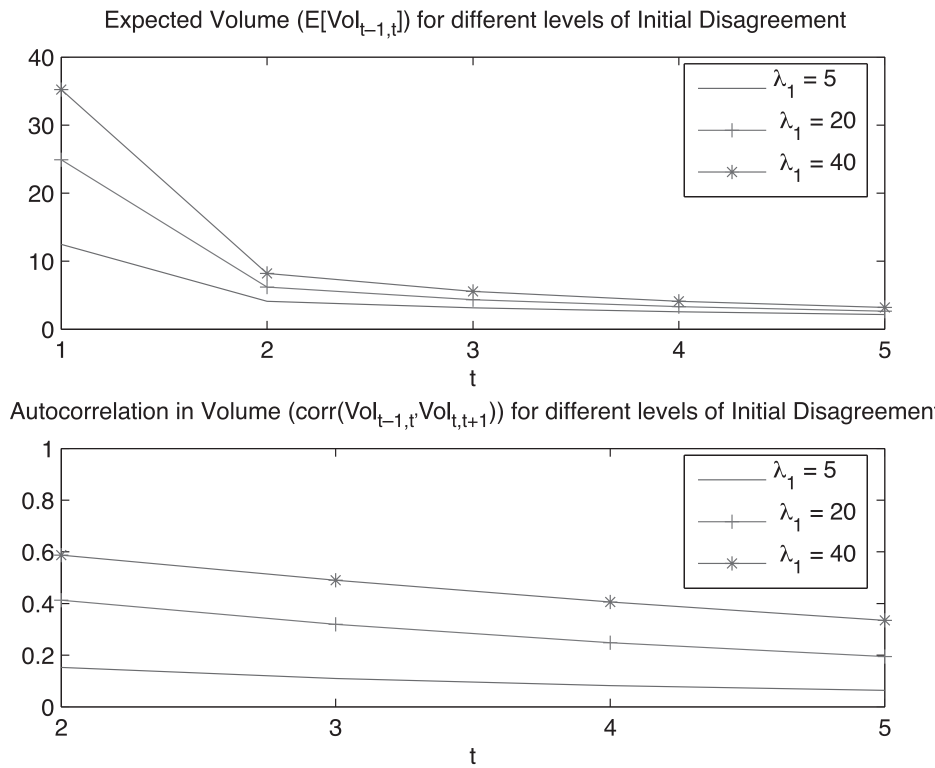

Figure 3. This figure shows the time-series dynamics of expected volume and volume autocorrelation after an initial jump in λ_1. Other model parameters are set to the following: $\rho_0 = 1,\ q_t = 0.1, \lambda_t = 1$, and $\delta = 1$.

convergence in beliefs and in turn to the observed exponential decay in volume. Also, note that the effect of high λ_1 on volume autocorrelation has the largest magnitude as compared to the effects of the other parameters.

It is difficult to interpret the time-series properties of volume since it is nonstationary. However, we confirm our intuition about the decay in volume and autocorrelation graphically in Figure 3. The time series of expected volume and volume autocorrelation are plotted for different initial levels of disagreement (λ_1). The effect of large disagreement in the initial period is quite persistent, leading to high expected volume and correlation over a number of periods.

The numerical examples suggest that, indeed, a pattern of occasional large disagreements followed by learning is an important feature in generating positive correlation in volume. Furthermore, higher overall levels of precision of the public signal lead to higher levels and correlations in volume. In Section IV, we are able to present similar results more formally in an infinite horizon model. In particular, we show that larger disagreement shocks lead to higher volume and higher autocorrelation in volume. We also show that while the level of volume increases in the frequency of disagreement shocks, autocorrelation in volume is lower for extremely frequent and extremely rare shocks than it is for an intermediate frequency of shocks.

B. Volume and Volatility

Next, we look at the correlation between volume and returns, or equivalently, price changes. Price changes in our model are given by

$$P_{t+1} - P_t = \bar{v}_{t+1} - \bar{v}_t = \pi_t(s_{t+1} - \bar{e}_{t+1} - \bar{v}_t). \tag{19}$$

As one would expect from a symmetric setup, there is no correlation between (signed) returns and volume, that is,

$$\mathrm{cov}(\mathrm{Vol}_{t+1}, P_{t+1} - P_t) = \mathrm{cov}(|x_{i,t+1} - x_{i,t}|, P_{t+1} - P_t) = 0. \tag{20}$$

This is because price changes are driven by the sequence of public signals $\{s_t\}$ and aggregate interpretations $\{\bar{e}_t\}$ whereas volume is driven by differences in interpretation $\{\Delta e_t\}$, which are independent of the s_t and $\bar{e}_t$. One could introduce a positive correlation between returns and volume either by introducing a component of trading driven by asymmetric information (as in standard RE models), or by introducing trading frictions (e.g., costly short selling) that introduce an asymmetry more directly. However, in order to keep the model tractable, we do not extend the model along these dimensions. Rather, we focus on the more robust feature of the data, namely, the correlation between volume and absolute returns.

Our analysis suggests another reason why occasional large disagreements followed by periods of relative agreement may be important characteristics of belief dynamics. Not only does this pattern generate higher levels of volume autocorrelation, but it also is an important factor in generating the positive correlation between volume and absolute returns that is empirically documented. This is because expected absolute returns depend on the variance in price changes:

$$E[|P_{t+1} - P_t|] = \sqrt{\frac{2}{\pi}\mathrm{var}(P_{t+1} - P_t)} \quad \text{where}$$
$$\mathrm{var}(P_{t+1} - P_t) = \pi_t^2(\mathrm{var}(s_{t+1}) + \mathrm{var}(\bar{v}_t) + \mathrm{var}(\bar{e}_{t+1})). \tag{21}$$

In particular, the volatility in price changes between dates t and $t+1$ depends on the variance of the average interpretation at date $t+1$ (i.e., $\mathrm{var}(\bar{e}_{t+1})$). Recall that the expected volume depends on the variance of $x_{i,t+1} - x_{i,t}$, which in turn depends on the variance of the differences in interpretation (i.e., $\mathrm{var}(\Delta e_{t+1})$). Both $\mathrm{var}(\bar{e}_{t+1})$ and $\mathrm{var}(\Delta e_{t+1})$ are given by $\lambda_{t+1}/2$, which implies that periods in which disagreement is high will have higher expected volume and higher absolute price changes. Intuitively, periods of higher disagreement lead not only to more trade, but also to higher price volatility. This implies that time-series variation in the disagreement process (i.e., in λ_{t+1}) leads to time-series correlation between price volatility and volume, despite the fact that within periods the two are uncorrelated.

IV. Infinite Horizon Model and Empirical Predictions

In this section, we develop a variant of our model that is better suited to generate empirical predictions. A limitation of the finite horizon model described in Section II is that the equilibrium is not stationary and there is a strong time trend. This trend makes it difficult to derive some of the comparative statics results analytically. In this section, we generate predictions based on an infinite horizon model in which investors are assumed to be myopic. This behavior can be justified using a specific overlapping generations model, but is made primarily for tractability. In particular, it allows us to ignore hedging demands, and so we are able to derive the equilibrium in closed form and also provide analytic proofs. Still, as we show, this simple setup captures the essence of the fully dynamic model from the earlier section. Volume has the same form as before, which suggests that predictions derived in this setup are also valid in the less tractable model from Section II.

The basic setup of the infinite horizon model is based on our earlier finite horizon setup. In this model, there are two assets: a risk-free asset that pays a gross return of $R > 1$, and a risky asset that pays dividends D_{t+1} at time $t+1$. The distribution of dividends is given by

$$D_{t+1} = F_{t+1} + d_{t+1}, \quad \text{where } d_{t+1} \sim N(0, \delta), \tag{22}$$

and the mean dividend process F_{t+1} is unobservable and given by

$$F_{t+1} = \alpha F_t + f_{t+1} \quad \text{where } f_{t+1} \sim N(0, \theta). \tag{23}$$

As before, there are two investors indexed by $i \in 1, 2$. Investors maximize utility over next period's wealth:

$$x_{i,t} = \arg\max_x E_{i,t}[-\exp\{-x(P_{t+1} + D_{t+1} - RP_t)\}]. \tag{24}$$

The rest of the model setup is similar to that of the finite horizon model. In addition to the dividend process D_t, investors also observe a public signal s_t about the mean dividend process

$$s_t = F_t + \varepsilon_t \quad \text{where } \varepsilon_t \sim N(e_{i,t}, q) \quad \text{and } e_{i,t} \sim N(0, \lambda_t). \tag{25}$$

The degree of disagreement is given by λ_t. We assume that agents generally agree on the interpretation but periodically have disagreements:

$$\lambda_1 = \lambda_{\tau+1} = \lambda_{2\tau+1} = \ldots \lambda^* \quad \text{and } \lambda_s = 0 \quad \text{for all other } s. \tag{26}$$

This allows us to derive predictions not just about the level of the disagreement shock (i.e., λ^*), but also about the frequency of disagreement shocks (i.e., $1/\tau$).[12]

[12]We can easily allow for investors to disagree on average—in fact, the proofs in the Appendix correspond to the general case of $\lambda_s = \lambda$.

To maintain notational consistency, we denote investor i's beliefs about F_{t+1} at time t by

$$F_{t+1} \sim N(v_{i,t}, \rho_t). \tag{27}$$

The evolution of this belief process is analogous to that in the finite horizon model, and is given by

$$v_{i,t+1} = \alpha[\pi_{v,t} v_{i,t} + \pi_{d,t} D_{t+1} + \pi_{s,t}(s_{t+1} - e_{i,t+1})] \quad \text{and} \quad \rho_{t+1} = \alpha^2 \rho_t \pi_{v,t} + \theta, \tag{28}$$

where $\pi_{d,t}$ and $\pi_{s,t}$ are the projection coefficients for D_{t+1} and s_{t+1}, and $\pi_{v,t} = 1 - \pi_{d,t} - \pi_{s,t}$. Note that persistence in the mean dividend process (i.e., $\alpha > 0$) is important in this infinite horizon version of the model, since we assume that investors are myopic. If $\alpha = 0$, the mean dividend process would be independent over time and current public signals would be uninformative about future dividends; as a result, investors' interpretations of the public signals would have no effect on volume dynamics.

The stationary linear equilibrium in this model is characterized by the following lemma:

LEMMA 2: *In the stationary linear equilibrium, investor i's optimal demand and price at date t are given by*

$$x_{i,t} = \phi_t \Delta v_{i,t} \quad \textit{and} \quad P_t = \frac{1}{R - \alpha} \bar{v}_t, \tag{29}$$

where ϕ_t is described in the Appendix.

This stationary linear equilibrium is similar to the equilibrium of the finite horizon economy described in Lemma 1. In particular, prices are linear in the average belief about mean dividend growth, and investor i's position depends on the disagreement between the two investors. Signed trade in this model also has a familiar form:

$$x_{i,t+1} - x_{i,t} = \underbrace{(\alpha \pi_v \phi_{t+1} - \phi_t) \Delta v_{i,t}}_{\text{belief-convergence term}} - \underbrace{\alpha \pi_s \phi_{t+1} \Delta e_{i,t+1}}_{\text{idiosyncratic term}}. \tag{30}$$

As in the finite horizon model, volume depends on two factors. The belief-convergence term is driven by prior differences in beliefs while the idiosyncratic term is due to different interpretations of the current period's public signal. As before, serial correlation in volume will be determined by the relative size of the learning component. From the proof of Lemma 2 we know that $\phi_t = \phi$ when there is no disagreement jump in period $t + 1$, and is smaller when there is a jump. In periods in which there are no disagreements (i.e., $\lambda_{t+1} = 0$), the idiosyncratic term is also zero and volume is driven by the learning term as beliefs converge. Formally, when $\lambda_{t+1} = 0$, we know that

$$\alpha \pi_v \phi_{t+1} - \phi_t < 0$$

and so volume is due to a convergence of beliefs. However, when there is a shock in disagreement, $\Delta e_{i,t+1}$ is not zero and both the idiosyncratic and the learning components of trade drive volume.

Based on the above characterization of the equilibrium, we derive the following predictions of the model based on the level and frequency of disagreement shocks.

PROPOSITION 3: *Suppose there is a jump in disagreement at date $t+1$, that is, $\lambda_{t+1} = \lambda^*$. Then:*

- *The return volatility at date $t+1$ is increasing in the size of the disagreement shock λ^*. Furthermore, if λ^* is large enough (as described in the Appendix), then expected volume at date $t+1$ and autocovariance in volume between dates $t+1$ and $t+2$ are increasing in λ_{t+1}. Otherwise, expected volume and autocovariance in volume are decreasing in λ_{t+1}.*
- *Price volatility and expected volume increase in the frequency of disagreement shocks, that is, the average over the τ periods of price volatility and expected volume decreases with τ. When the disagreement shock λ^* is large enough, then volume autocorrelation is nonmonotonic in the frequency of disagreement shocks. In particular, the average volume autocorrelation over τ periods is first increasing and then decreasing in τ.*
- *Suppose the aggregate supply of the risky asset is given by $Q > 0$. Then the expected return on the asset $E(R_{t+1})$ is increasing in the magnitude of the disagreement shock λ^*. Moreover, if there is a jump in disagreement every τ periods, then expected returns increase in the frequency of disagreement shocks.*

The intuition for these results is as follows. The larger the disagreement shock λ^* at date $t+1$, the higher the price uncertainty at date $t+1$, and as a result the higher the price volatility. The relationship between disagreement and volume characteristics is more complicated as there are two potentially offsetting effects of a jump in disagreement. Intuitively, a large jump in disagreement leads to volume in the current period and volume in future periods. However, the anticipation of a large jump in disagreement in the next period leads investors to hold smaller positions in the current period because of higher future uncertainty. Specifically, the first effect of a higher λ_{t+1} is larger idiosyncratic trade at date $t+1$, and so higher convergence trade at date $t+2$. The second effect of a higher λ_{t+1} is due to investors taking less aggressive positions at date t (i.e., lower ϕ_t) because of higher future uncertainty. When λ_{t+1} is not too large (i.e., when $\alpha\pi_v\phi - \phi_t > 0$), the effect of a lower ϕ_t is to reduce the convergence component of volume. But when λ_{t+1} is large enough (i.e., if $\alpha\pi_v\phi - \phi_t \leq 0$), increasing λ_{t+1} decreases ϕ_t, which increases the convergence component of volume and so expected volume and autocorrelation in volume both increase with λ_{t+1}. This relationship between the jump in disagreement, λ^*, and the level and autocorrelation in volume confirms our intuition from the plots in the bottom panel of Figure 2.

Moreover, note that for a given level of the disagreement shock λ^*, increasing the frequency of shocks (i.e., decreasing τ) increases the average price volatility and expected volume over these τ periods. When the disagreement shock is large (i.e., $\alpha\pi_v\phi_{t+1} - \phi_t \leq 0$), extremely frequent disagreement shocks actually lead to a decrease in volume autocorrelation since the idiosyncratic component of volume dominates the learning component. Figure 4 shows an instance of these results when the steady-state disagreement level is not zero. Again, increasing τ decreases price volatility (as measured by absolute returns), expected volume, and serial correlation in volume. However, since the underlying disagreement shock is large, when disagreement shocks are extremely frequent (e.g., $\tau = 2, 3$), volume autocorrelation is lower than when they are less frequent. This confirms our intuition from the finite horizon model in which we argue that large, relatively infrequent disagreements lead to high autocorrelation in volume.

As our primary focus in the paper is on the relationship between disagreement and volume and return dynamics, we have assumed that the aggregate supply of the risky asset is zero. If the aggregate supply of the asset is a positive constant Q instead, then the expected return of the model in steady-state equilibrium is given by

$$E(R_{t+1}) = \mathrm{var}_{i,t}[P_{t+1} + D_{t+1} - RP_t]Q, \tag{31}$$

where $\mathrm{var}_{i,t}[P_{t+1} + D_{t+1} - RP_t]$ is the variance of payoffs conditional on the investors' time t information. Note that since the investors have symmetric information sets, the conditional variance is the same across both investors. From the proof of Lemma 2, we know that the conditional variance $\mathrm{var}_{i,t}[P_{t+1} + D_{t+1} - RP_t]$ is linear in the disagreement λ_{t+1} at date $t+1$. This is because a higher level of disagreement at date $t+1$ implies that for investors at date t, the payoff is more uncertain. This immediately implies the comparative statics results in Proposition 3.

In particular, this model suggests that expected returns increase with the level of disagreement. Note that this is in contrast to other DO models (e.g., Miller (1977)), which predict a negative relationship between the two. The reason for the positive relationship in our model is that a higher level of disagreement in the future leads to more uncertainty in payoffs today, and this increase in risk leads investors to require a higher expected return. Moreover, the relationship between expected returns and disagreement is empirically unclear. While some papers claim to document a negative relationship (e.g., Diether, Malloy, and Scherbina (2002) and Hong and Stein (2003)), others find a positive relationship between the two (e.g., Qu, Starks, and Yan (2004) and Banerjee (2010)). Also, Ball and Kothari (1991) document a spike in abnormal returns on the date of the announcement, which is consistent with the predictions of our model. The positive relationship between disagreement, volume, and expected returns is also consistent with the high-volume return premium documented by Gervais, Kaniel, and Mingelgrin (2001).

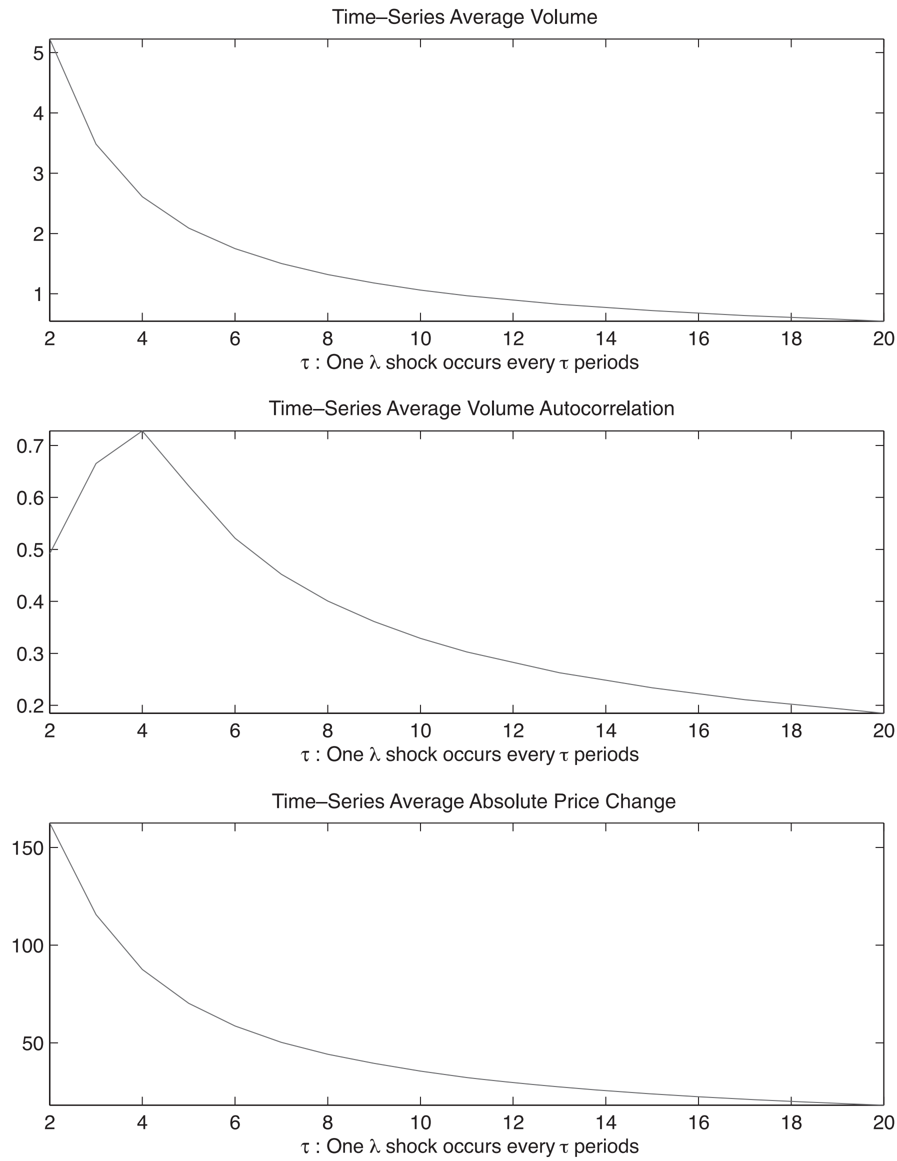

Figure 4. This figure depicts the effect of increasing the frequency of the disagreement shock on average volume, autocorrelation in volume, and average absolute price change.

A. Empirical Predictions about Volume, Volatility, and Returns

The results developed in the previous section can be used to generate a number of empirically testable predictions. Since disagreement across investors is a latent variable, we need to construct a proxy for it. In our model, when there is a large jump in disagreement and a corresponding large jump in volume, volume and disagreement are positively correlated. This implies that, based on Proposition 3, we have the following empirical prediction.

PREDICTION 1: *When abnormal volume is high, volume autocorrelation, volatility, and mean returns increase with abnormal volume.*

We present a preliminary test of this prediction. We use the daily turnover of a stock (traded volume scaled by market value of equity) as our measure of volume. For each stock in CRSP from 1980 onwards, we calculate abnormal daily turnover relative to the mean turnover for the month, scaled by the standard deviation of the turnover for that year. We then sort these firms into 10 bins based on their scaled abnormal volume. We report the mean autocorrelation in volume, the mean volatility in returns, and the mean returns over the next 10 days for each bin across all firms in Figure 5. The evidence is consistent with our predictions. When the abnormal volume is high, volume autocorrelation, return volatility, and mean returns are all increasing in volume.

Another prediction of the model follows from the fact that in our model, return volatility is always increasing in disagreement. Hence, on average, conditioning on whether we have high or low return volatility should allow us to distinguish between high and low disagreement. As a result, we expect to find the following.

PREDICTION 2: *When return volatility is high, autocorrelation in volume is increasing in volume. When return volatility is low, volume autocorrelation is decreasing in volume.*

Figure 6 shows preliminary evidence consistent with this prediction. Specifically, using our sample of CRSP firms, we sort observations into deciles based on return volatility and plot the relationship between volume autocorrelation across abnormal volume bins for the lowest and highest decile of return volatility. As predicted, for the decile of observations with high volatility, autocorrelation in volume is increasing in abnormal volume. In contrast, for the low volatility decile, autocorrelation appears to decrease with abnormal volume. This is consistent with the predictions in Proposition 3 since low return volatility implies low disagreement (low λ), and this implies in turn that autocorrelation in volume is decreasing in disagreement.

The results of Proposition 3 can also be used to design event-based empirical tests of the model. In particular, these predictions can be used to study volume and return characteristics around information events such as earnings announcements, which are likely to be associated with large jumps in disagreement. For instance, if investors exhibit more disagreement on earnings announcement days relative to other days, Proposition 3 suggests we should observe the following:

PREDICTION 3: *Days with earnings announcements are associated with higher levels of volume and volatility and are followed by higher volume autocorrelation compared to days without earnings announcements.*

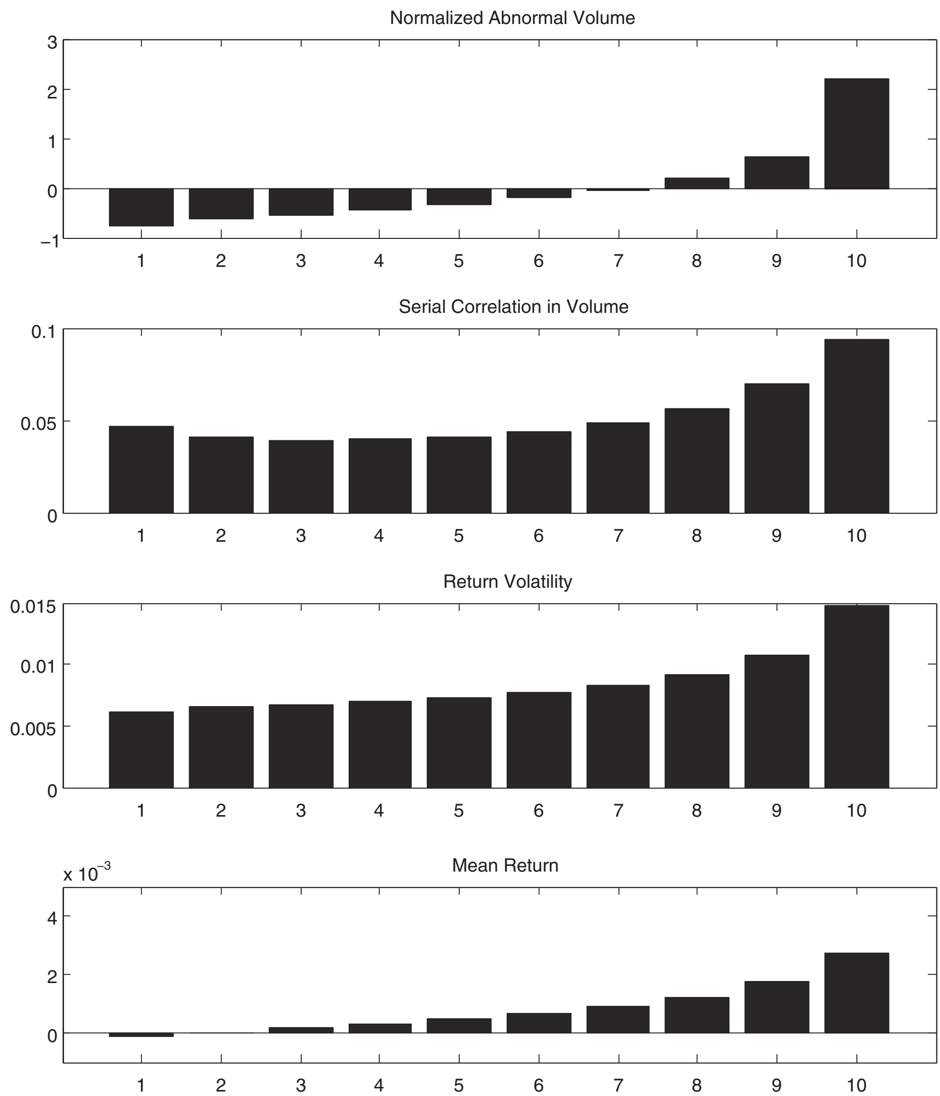

Figure 5. This figure presents the empirical distribution of normalized abnormal volume sorted into deciles and the corresponding levels of autocorrelation in volume, return volatility, and mean returns across these deciles.

Some empirical evidence is consistent with these results. For instance, Atiase and Bamber (1994) show that volume around earnings announcements is increasing in the level of predisclosure analyst forecast dispersion. Chae (2005) documents that volume jumps on an earnings announcement and then gradually decays over the next few days, and Ball and Kothari (1991) document that return volatility spikes on the date of an earnings announcement, but decays rapidly afterwards. However, our model's predictions can be made finer, since

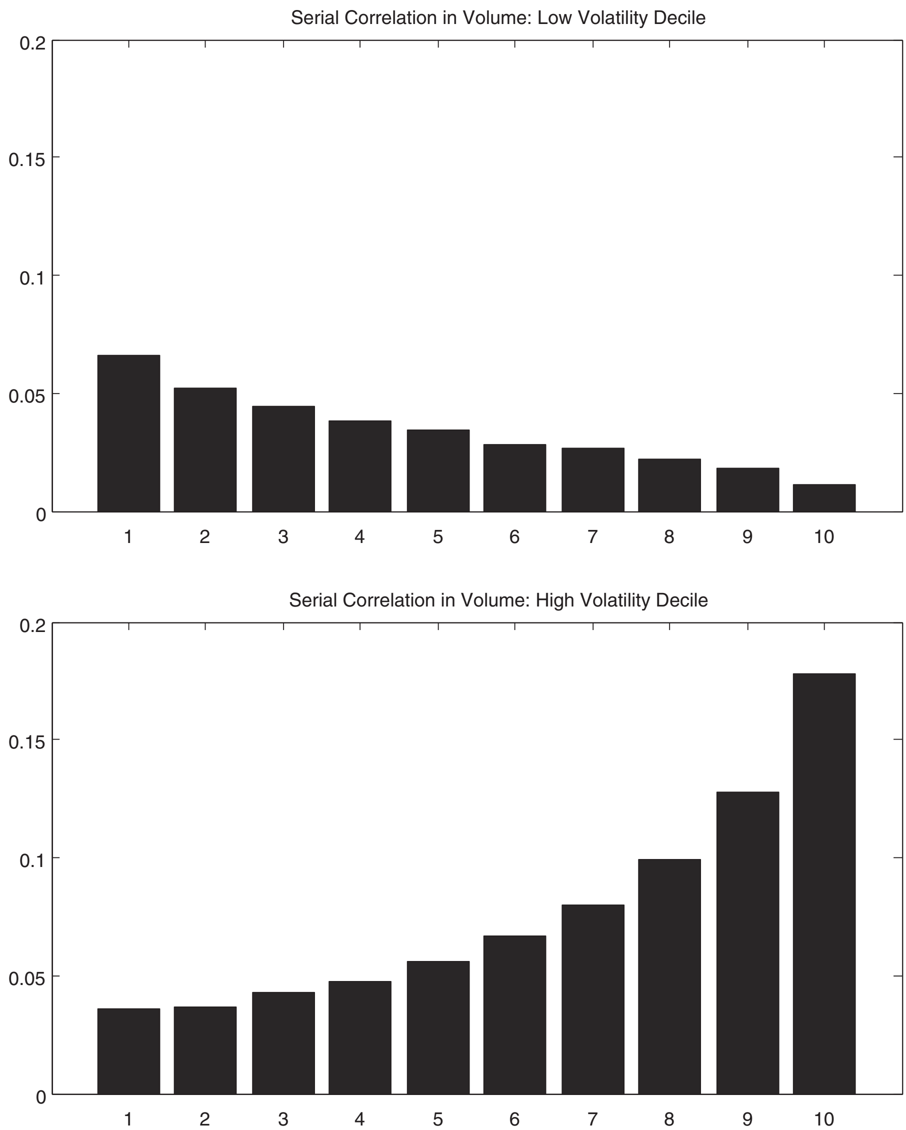

Figure 6. This figure depicts the empirical distribution of autocorrelation in volume conditional on abnormal volume for low return volatility and high return volatility deciles.

they relate the size of the disagreement shock to these return and volume characteristics, and can potentially be used to identify which events lead to more disagreement among investors, and which lead to less. Moreover, Proposition 3 provides novel predictions relating return and volume characteristics with the frequency of disagreement that have not been tested to the best of our knowledge.

V. Conclusions

We develop a dynamic differences of opinion model to analyze the relationship between disagreement and trading volume around public announcements. We show that infrequent but major disagreements among agents lead to patterns in volume and returns that are empirically observed. In particular, such disagreements lead to volume clustering, even when there is no persistence in fundamentals, and to time-series correlation between volume and volatility. We also develop new predictions that relate the size and frequency of disagreement shocks, volume autocorrelation, and return volatility, and find that the preliminary empirical evidence is consistent with these predictions.

The differences of opinion framework is an interesting and promising alternative to the standard asymmetric information models. The view that agents may "agree to disagree" not only seems plausible, but also appears better suited to address some of the empirical evidence involving trading volume. The current model provides a simple and tractable benchmark that can be extended in a number of interesting ways. For instance, in a differences of opinion setting, it would be interesting to study the effects of trading costs and restrictions (e.g., short sales constraints) on volume and return dynamics, the role of different types of information (e.g., "soft" vs. "hard") in generating disagreement, volume, and return patterns around events when uncertainty and disagreement may be higher (e.g., initial public offerings or takeover/merger decisions), and the effects of learning about, and speculating on, disagreement over time.

Appendix: Proofs

Proof for Lemma 1: We will prove this lemma by induction.

Base Step: At date $t = T - 1$, we know that the above holds, since

$$x_{i,T-1} = \frac{v_{i,T-1} - P_{T-1}}{\rho_{T-1} + \delta}, \quad P_{T-1} = \bar{v}_{T-1} \quad \text{and}$$
$$EU_{i,T-1} = -\exp\left\{-\frac{(v_{i,T-1} - P_{T-1})^2}{2(\rho_{T-1} + \delta)}\right\}. \tag{A1}$$

In particular, this implies that $\phi_{T-1} = \frac{1}{K_{T-1}} = \frac{1}{\rho_{T-1}+\delta}$.

Iterative Step: Suppose the conjecture that holds for all $\tau > t$. We show it therefore holds for t. Note that the beliefs of investor i can be written as

$$s_{t+1} - e_{i,t+1} = V + \theta_{i,t+1}$$
$$s_{t+1} - \bar{e}_{t+1} = V + \theta_{i,t+1} + \frac{1}{2}(e_{i,t+1} - e_{j,t+1}),$$

where $\theta_{i,t+1} \sim N(0, q_{t+1})$. This implies that investor i's beliefs about $v_{i,t+1}$ and P_{t+1} are given by

$$
\begin{aligned}
Z &\equiv \begin{pmatrix} P_{t+1} \\ v_{i,t+1} \end{pmatrix} \\
&= N\left(\begin{pmatrix} (1-\pi_t)\bar{v}_t + \pi_t v_{i,t} \\ v_{i,t} \end{pmatrix}, \begin{pmatrix} \pi_t^2 \left(\rho_t + q_{t+1} + \frac{1}{2}\lambda_{t+1}\right) & \pi_t^2(\rho_t + q_{t+1}) \\ \pi_t^2(\rho_t + q_{t+1}) & \pi_t^2(\rho_t + q_{t+1}) \end{pmatrix} \right) \\
&= N\left(\begin{pmatrix} m_{i,t} \\ v_{i,t} \end{pmatrix}, \begin{pmatrix} \eta_t & \pi_t\rho_t \\ \pi_t\rho_t & \pi_t\rho_t \end{pmatrix} \right).
\end{aligned} \tag{A2}
$$

At date t, investor i's problem is given by

$$
\begin{aligned}
x_{it} &= \arg\max_x E_{i,t}[-\exp\{-x(P_{t+1} - P_t) \\
&\qquad\qquad - x_{i,t+1}(P_{t+2} - P_{t+1}) \ldots - x_{i,T-1}(F - P_{T-1})\}] \\
&= \arg\max_x E_{i,t}\left[-\exp\left\{ -x(P_{t+1} - P_t) - \frac{1}{2K_{t+1}}(v_{i,t+1} - P_{t+1})^2 \right\} \right] \\
&= \arg\max_x E_{i.t}[-\exp\{c + b'Z + Z'AZ\}] \\
&= \arg\max_x \exp\left\{ -\frac{1}{2}\left(\mu_Z'\Sigma_Z^{-1}\mu_Z - 2c\right) \right. \\
&\qquad\qquad \left. + \frac{1}{2}(\mu_Z + \Sigma_Z b)'(I - 2A\Sigma_Z)^{-1}\Sigma_Z^{-1}(\mu_Z + \Sigma_Z b) \right\},
\end{aligned}
$$

where $c = xP_t$, $b' = (-x, 0)$, and $A = -\frac{1}{2K_{t+1}}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. The relevant first-order condition is given by

$$
\frac{\partial c}{\partial x} + (\mu_Z + \Sigma_Z b)'(I - 2A\Sigma_Z)^{-1}\frac{\partial b}{\partial x} = 0, \tag{A3}
$$

which reduces to

$$
(m_{i,t} - P_t - x\eta_t)K_{t+1} + (v_{i,t} - P_t - x\pi_t\rho_t)(\eta_t - \pi_t\rho_t) = 0. \tag{A4}
$$

Solving for x, we get

$$
x_{i,t} = \frac{K_{t+1}(m_{i,t} - P_t) + (\eta_t - \pi_t\rho_t)(v_{i,t} - P_t)}{K_{t+1}\eta_t + (\pi_t\rho_t)(\eta_t - \pi_t\rho_t)}. \tag{A5}
$$

Next, aggregating over all investors, we have

$$
\int x_{i,t} = 0 \quad \Rightarrow \quad P_t = \bar{v}_t, \tag{A6}
$$

which implies that

$$m_{i,t} - P_t = \pi_t(v_{i,t} - P_t).$$

This implies that

$$x_{i,t} = \frac{K_{t+1}\pi_t + (\eta_t - \pi_t\rho_t)}{K_{t+1}\eta_t + \pi_t\rho_t(\eta_t - \pi_t\rho_t)}(v_{i,t} - P_t) = \phi_t(v_{i,t} - P_t). \tag{A7}$$

Finally, the expected utility at time t is proportional to

$$\begin{aligned} EU_t &\propto \exp\left\{-\frac{1}{2}\left(\mu_Z'\Sigma_Z^{-1}\mu_Z - 2c\right) + \frac{1}{2}(\mu_Z + \Sigma_Z b)'(I - 2A\Sigma_Z)^{-1}\Sigma_Z^{-1}(\mu_Z + \Sigma_Z b)\right\} \\ &= \exp\left\{-\frac{1}{2}\frac{\left(K_{t+1}\pi_t^2 + \pi_t\rho_t(1-\pi_t)^2 + (\eta_t - \pi_t\rho_t)\right)}{K_{t+1}\eta_t + \pi_t\rho_t(\eta_t - \pi_t\rho_t)}(v_{i,t} - P_t)^2\right\}, \end{aligned} \tag{A8}$$

which implies that

$$K_t = \frac{K_{t+1}\eta_t + \pi_t\rho_t(\eta_t - \pi_t\rho_t)}{K_{t+1}\pi_t^2 + \pi_t\rho_t(1-\pi_t)^2 + (\eta_t - \pi_t\rho_t)}. \tag{A9}$$

This completes the iterative step.

We know that $K_{T-1} = \rho_{T-1} + \delta$. Suppose $K_{t+1} = \rho_{t+1} + \gamma_{t+1}$. Then,

$$\begin{aligned} K_t &= \frac{(\rho_{t+1} + \gamma_{t+1})\,\eta_t + \pi_t\rho_t(\eta_t - \pi_t\rho_t)}{(\rho_{t+1} + \gamma_{t+1})\pi_t^2 + \pi_t\rho_t(1-\pi_t)^2 + (\eta_t - \pi_t\rho_t)} \\ &= \frac{\gamma_{t+1}\eta_t + \rho_t\left(\eta_t - \pi_t^2\rho_t\right)}{\gamma_{t+1}\pi_t^2 + \left(\eta_t - \pi_t^2\rho_t\right)} = \rho_t + \frac{\gamma_{t+1}\left(\eta_t - \pi_t^2\rho_t\right)}{\gamma_{t+1}\pi_t^2 + \left(\eta_t - \pi_t^2\rho_t\right)} \\ &= \rho_t + \frac{\gamma_{t+1}(q_{t+1} + \lambda_{t+1}/2)}{\gamma_{t+1} + (q_{t+1} + \lambda_{t+1}/2)} = \rho_t + \gamma_t, \end{aligned}$$

where

$$\frac{1}{\gamma_t} = \frac{1}{\gamma_{t+1}} + \frac{1}{q_{t+1} + \lambda_{t+1}/2}. \tag{A10}$$

This also implies that

$$\phi_t = \frac{1}{\rho_t}\left(\frac{\rho_t + \gamma_t\left(\dfrac{q_{t+1}}{q_{t+1} + \lambda_{t+1}/2}\right)}{\rho_t + \gamma_t}\right), \tag{A11}$$

which completes the proof. Q.E.D.

Proof of Corollary 1: Note that if $\delta = 0$, then $\gamma_t = 0$ for all t and so $K_t = \rho_t$. This also implies that $\phi_t = \frac{1}{\rho_t}$. Similarly, when $\lambda_t = 0$, $\eta_t = \rho_t\pi_t$, which implies $x_{i,t} = \frac{1}{\rho_t}(v_{i,t} - P_t)$. Q.E.D.

Proof of Proposition 1: The expression for volume follows immediately from the expression for the signed trade $T_{i,t+1}$. Since $\phi_t > 0$ and $\pi_t \geq 0$, we know that $-\phi_{t+1}\pi_t \leq 0$. When $\lambda_t = \lambda$ and $q_t = q$ for all t, then

$$\phi_{t+1}(1-\pi_t) - \phi_t = \frac{1}{\rho_t}\left(\frac{\rho_{t+1} + \gamma_{t+1}\left(\dfrac{q}{q+\lambda/2}\right)}{\rho_{t+1}+\gamma_{t+1}} - \frac{\rho_t + \gamma_t\left(\dfrac{q}{q+\lambda/2}\right)}{\rho_t+\gamma_t}\right)$$

$$= \frac{1}{\rho_t}\left(\frac{(\rho_t(\gamma_t - \gamma_{t+1}) - \rho_t\pi_t\gamma_t)\left(1 - \dfrac{q}{q+\lambda/2}\right)}{(\rho_{t+1}+\gamma_{t+1})(\rho_t+\gamma_t)}\right) \leq 0$$

since $\gamma_t \leq \gamma_{t+1}$. Q.E.D.

Proof of Proposition 2: Since $T_{i,t+1}$ is normally distributed, and $\mathrm{Vol}_{t+1} = |T_{i,t+1}|$, volume has a half-normal distribution. Specifically, this implies that

$$E[|\mathrm{Vol}_{t+1}|] = E[|T_{i,t+1}|] = \sqrt{\frac{2}{\pi}\sigma^2_{T_{i,t+1}}}$$

$$\mathrm{corr}(\mathrm{Vol}_{t+2}, \mathrm{Vol}_{t+1}) = \frac{2}{\pi-2}\left((1-\rho^2)^{3/2} - 1 + \rho^2\sqrt{1-\rho^2} + |\rho|\arctan\left(\frac{|\rho|}{\sqrt{1-\rho^2}}\right)\right),$$

where $\rho = \frac{\mathrm{cov}(T_{i,t+1}, T_{i,t+2})}{\sqrt{\mathrm{var}(T_{i,t+1})\mathrm{var}(T_{i,t+2})}}$. The variance and covariance of signed trade are given by

$$\mathrm{var}[x_{i,t+1} - x_{i,t}] = (\phi_{t+1}(1-\pi_t) - \phi_t)^2\mathrm{var}[\Delta v_{i,t}] + \phi_{t+1}^2\pi_t^2\mathrm{var}[\Delta e_{i,t+1}]$$

$$\begin{aligned}&\mathrm{cov}(x_{i,t+2} - x_{i,t+1}, x_{i,t+1} - x_{i,t})\\ &= (\phi_{t+2}(1-\pi_{t+1}) - \phi_{t+1})\\ &\quad\times\left\{(\phi_{t+1}(1-\pi_t) - \phi_t)(1-\pi_t)\mathrm{var}[\Delta v_{i,t}] - \phi_{t+1}\pi_t^2\lambda_{t+1}/2\right\},\end{aligned}$$

and the variance in the difference in valuations is defined recursively as

$$\mathrm{var}[\Delta v_{i,0}] = \frac{1}{4}\sigma_0 \quad \text{and} \quad \mathrm{var}[\Delta v_{i,t+1}] = (1-\pi_t)^2\mathrm{var}[\Delta v_{i,t}] + \pi_t^2\lambda_{t+1}/2$$

since, in general, we assume that $x_{i,0} - x_{j,0} \sim N(0, \sigma_0)$. Q.E.D.

Proof of Lemma 2: Conjecture a linear equilibrium with the price given by

$$P_t = A_t\bar{v}_t. \tag{A12}$$

Given the price conjecture, investor i's beliefs about dollar returns are given by

$$E_{i,t}[P_{t+1} + D_{t+1} - RP_t] = A_{t+1}\alpha(\pi_{v,t}\bar{v}_{i,t} + (\pi_{s,t} + \pi_{d,t})v_{i,t}) + v_{i,t} - RP_t \tag{A13}$$

$$\mathrm{var}_{i,t}[P_{t+1}+D_{t+1}-RP_t]=(A_{t+1}\alpha(\pi_{s,t}+\pi_{d,t})+1)^2\rho_t+(A_{t+1}\alpha\pi_{d,t}+1)^2\delta$$
$$+(A_{t+1}\alpha\pi_{s,t})^2(q+\lambda_{t+1}/2)\equiv V_{R,t}, \quad \text{(A14)}$$

where the projection coefficients are given by

$$\pi_{d,t}=\frac{1/\delta}{1/\delta+1/\rho_t+1/q},\quad \pi_{s,t}=\frac{1/q}{1/\delta+1/\rho_t+1/q},\quad \pi_{v,t}=1-\pi_{s,t}-\pi_{d,t}. \quad \text{(A15)}$$

Market clearing implies that

$$\int_i \frac{E_{i,t}[P_{t+1}+D_{t+1}-RP_t]}{\mathrm{var}_{i,t}[P_{t+1}+D_{t+1}-RP_t]}=0 \quad\Rightarrow\quad (A_{t+1}\alpha+1)\bar{v}_t=RP_t, \quad \text{(A16)}$$

which implies $A_t=\frac{1}{R}(1+\alpha A_t)$ and the optimal demand is given by

$$x_{i,t}=\phi_t\Delta v_{i,t}\quad \text{where}\ \ \phi_t=\frac{1}{V_{R,t}}(A_{t+1}\alpha(\pi_{s,t}+\pi_{d,t})+1). \quad \text{(A17)}$$

The steady state of the equilibrium is characterized by $\rho=\rho_t=\rho_{t+1}$, which solves

$$\rho=\alpha^2\rho\pi_v+\theta\Rightarrow\rho=\frac{\alpha^2}{1/\rho+1/\delta+1/q}+\theta, \quad \text{(A18)}$$

and $A_t=A_{t+1}=1/(R-\alpha)\equiv A$. Consequently,

$$\phi_t=\frac{1}{V_{R,t}}\,(1+\alpha A(\pi_s+\pi_d)) \quad \text{(A19)}$$

$$V_{R,t}=(1+\alpha A(\pi_s+\pi_d))^2\rho+(1+\alpha A\pi_d)^2\delta+(\alpha A\pi_s)^2(q+\lambda_{t+1}/2). \quad \text{(A20)}$$

Q.E.D.

Proof of Proposition 3: Level of Disagreement: Suppose there is a shock in λ at time $t+1$, that is, $\lambda_{t+1}=\lambda^*>\lambda_s=\lambda$ for all $s\neq t+1$. This implies that for all $s\neq t$, $V_{Q,t}>V_{Q,s}=V_Q$ and consequently $\phi_t<\phi_s=\phi$. In particular, for what follows note that

$$\frac{\phi_t}{\phi}\le\alpha\pi_v\Leftrightarrow\lambda^*\ge\lambda+\frac{2V_Q(1-\alpha\pi_v)}{\alpha\pi_v\,(A\alpha\pi_s)^2}. \quad \text{(A21)}$$

Also, note that the regression coefficients π_s, π_d, and π_v, the steady-state variance ρ, and the price coefficient A do not change from their steady state.

The price volatility at date $t+1$ depends on

$$\mathrm{var}\,(P_{t+1}-P_t)=A^2\left[\frac{2\alpha^2}{1+\alpha\pi_v}\left((\pi_d+\pi_s)^2\frac{\theta}{1-\alpha^2}+\pi_d^2\delta+\pi_s^2\,(q+\lambda_{t+1}/2)\right)\right],$$

which is increasing in λ_{t+1}. Signed trade between dates t and $t+1$ is given by

$$x_{i,t+1} - x_{i,t} = (\alpha\pi_v\phi_{t+1} - \phi_t)\,\Delta v_{i,t} - \alpha\pi_s\phi_{t+1}\Delta e_{i,t+1}. \tag{A22}$$

This implies that expected volume as a result of the disagreement shock is given by

$$\mathrm{var}(x_{i,t+1} - x_{i,t}) = (\alpha\pi_v\phi_{t+1} - \phi_t)^2\mathrm{var}(\Delta v_{i,t}) + (\alpha\pi_s\phi_{t+1})^2\lambda_{t+1}/2 \tag{A23}$$

$$= \frac{1}{2}\alpha^2\pi_s^2\phi^2\left[\lambda\frac{\left(\dfrac{\phi_t}{\phi} - \alpha\pi_v\right)^2}{\left(1-\alpha^2\pi_v^2\right)} + \lambda_{t+1}\right]. \tag{A24}$$

An increase in the size of the disagreement shock has two effects on the expected volume. The first effect is a direct increase in expected volume through the λ_{t+1} term. The second effect depends on the size of λ_{t+1}/λ—in particular, if $\phi_t(\lambda_{t+1}) > \alpha\pi_v\phi$, then an increase in λ_{t+1} decreases expected volume. But when $\phi_t(\lambda_{t+1}) \leq \alpha\pi_v\phi$, an increase in λ_{t+1} again decreases $\phi_t(\lambda_{t+1})$ and so increases expected volume. In particular, when λ_{t+1} is large enough so that $\frac{\phi_t}{\phi} - \alpha\pi_v \leq 0$, then an increase in λ_{t+1} increases expected volume.

The autocovariance in volume depends on the absolute value of the serial covariance in signed trade

$$\begin{aligned}&|\mathrm{cov}(x_{i,t+2} - x_{i,t+1}, x_{i,t+1} - x_{i,t})|\\ &\quad = \left|\alpha(\alpha\pi_v\phi_{t+2} - \phi_{t+1})\left[\pi_v(\alpha\pi_v\phi_{t+1} - \phi_t)\mathrm{var}(\Delta v_{i,t}) + \alpha\pi_s^2\phi_{t+1}\lambda_{t+1}/2\right]\right|\end{aligned} \tag{A25}$$

$$= \frac{1}{2}\alpha^2\phi^2\pi_s^2\lambda(1-\alpha\pi_v)\left|\frac{\alpha\pi_v\left(\dfrac{\phi_t}{\phi} - \alpha\pi_v\right)}{\left(1-\alpha^2\pi_v^2\right)} - \frac{\lambda_{t+1}}{\lambda}\right|. \tag{A26}$$

In this case, when λ_{t+1} is large enough so that $\phi_t - \alpha\pi_v\phi \leq 0$, increasing λ_{t+1} increases the autocovariance. On the other hand, when

$$\frac{\alpha\pi_v\left(\dfrac{\phi_t}{\phi} - \alpha\pi_v\right)}{\left(1-\alpha^2\pi_v^2\right)} - \frac{\lambda_{t+1}}{\lambda} > 0,$$

then increasing λ_{t+1} decreases the autocovariance in volume.

Frequency of Disagreement: Suppose the disagreement shock occurs at $t+1$. This implies that

$$\begin{aligned}\mathrm{var}(\Delta v_{i,t+1}) &= \alpha^2\pi_v^2\mathrm{var}(\Delta v_{i,t}) + \alpha^2\pi_s^2\lambda^*/2 \quad \text{and}\\ \mathrm{var}(\Delta v_{i,t+s+1}) &= \left(\alpha^2\pi_v^2\right)^s\mathrm{var}(\Delta v_{i,t+1}).\end{aligned} \tag{A27}$$

Moreover, in the steady state of this equilibrium, since there is a disagreement shock every τ periods, we know that

$$\mathrm{var}(\Delta v_{i,t+\tau}) = \mathrm{var}(\Delta v_{i,t}) = \left(\alpha^2\pi_v^2\right)^{\tau-1}\left(\alpha^2\pi_v^2\mathrm{var}(\Delta v_{i,t}) + \alpha^2\pi_s^2\lambda^*/2\right),$$

which implies

$$\mathrm{var}(\Delta v_{i,t}) = \frac{\left(\alpha^2\pi_v^2\right)^{\tau-1}\alpha^2\pi_s^2\lambda^*/2}{1-\left(\alpha^2\pi_v^2\right)^{\tau}} \equiv \sigma_\tau^2; \tag{A28}$$

note that $\sigma_\tau^2 \geq \sigma_{\tau'}^2$ for $\tau' > \tau$.

Volume at date $t+1$ depends on

$$\mathrm{var}(x_{i,t+1} - x_{i,t}) = (\alpha\pi_v\phi - \phi_t)^2\sigma_\tau^2 + (\alpha\pi_s\phi)^2\lambda^*/2$$

and volume on the following dates $t+2,\ldots,t+s$ depend on

$$\mathrm{var}(x_{i,t+s+1} - x_{i,t+s}) = \phi^2(1-\alpha\pi_v)^2\left(\alpha^2\pi_v^2\right)^{s}\left(\alpha^2\pi_v^2\sigma_\tau^2 + \alpha^2\pi_s^2\lambda^*/2\right). \tag{A29}$$

The average volume over τ periods is given by

$$\begin{aligned}\frac{1}{\tau}\sum_{s=0}^{\tau}E[|x_{i,t+s+1} - x_{i,t+s}|] &= \frac{1}{\tau}\sum_{s=1}^{\tau}\sqrt{\frac{2}{\pi}\mathrm{var}(x_{i,t+s+1} - x_{i,t+s})}\\ &\quad + \frac{1}{\tau}\sqrt{\frac{2}{\pi}\mathrm{var}(x_{i,s+1} - x_{i,s})},\end{aligned}$$

which implies that expected volume decreases as τ increases.

Similarly, when $\tau > 1$, then covariance between volume at $t+1$ and $t+2$ depends on

$$\begin{aligned}&|\mathrm{cov}(x_{i,t+2} - x_{i,t+1}, x_{i,t+1} - x_{i,t})|\\ &\quad = \left|\alpha(\alpha\pi_v\phi_{t+2} - \phi_{t+1})\left[\pi_v(\alpha\pi_v\phi_{t+1} - \phi_t)\mathrm{var}(\Delta v_{i,t}) + \alpha\pi_s^2\phi_{t+1}\lambda^*/2\right]\right|\\ &\quad = \left|\alpha\phi(\alpha\pi_v - 1)\left[\pi_v(\alpha\pi_v\phi - \phi_t)\sigma_\tau^2 + \alpha\pi_s^2\phi\lambda^*/2\right]\right|\end{aligned}$$

and on the following days, it depends on

$$\begin{aligned}&|\mathrm{cov}(x_{i,t+s+2} - x_{i,t+s+1}, x_{i,t+s+1} - x_{i,t+s})|\\ &\quad = \left|\alpha\phi\pi_v(\alpha\pi_v - 1)^2\left[\left(\alpha^2\pi_v^2\right)^{s-1}\left(\alpha^2\pi_v^2\sigma_\tau^2 + \alpha^2\pi_s^2\lambda^*/2\right)\right]\right|.\end{aligned}$$

When λ^* is small enough so that $\alpha\pi_v\phi - \phi_t > 0$, then increasing τ leads to a decrease in the average volume autocorrelation. However, when λ^* is large enough so that $\alpha\pi_v\phi - \phi_t < 0$, then decreasing τ will decrease the covariance in the first period (i.e., $\mathrm{cov}(x_{i,t+2} - x_{i,t+1}, x_{i,t+1} - x_{i,t})$), and consequently will decrease the average correlation if τ is small enough. However, when τ is large enough (and so σ_τ^2 is small enough), the remaining terms dominate, and the overall effect is to decrease the average volume autocorrelation with an increase in τ. Q.E.D.

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