Price Drift as an Outcome of Differences in Higher-Order Beliefs

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Motivated by the insight of Keynes (1936) on the importance of higher-order beliefs in financial markets, we examine the role of such beliefs in generating drift in asset prices. We show that in a dynamic setting, a higher-order difference of opinions is necessary for heterogeneous beliefs to generate price drift. Such drift does not arise in standard difference of opinion models, since investors' beliefs are assumed to be common knowledge. Our results stand in contrast to those of Allen, Morris, and Shin (2006) and others, as we argue that in rational expectation equilibria, heterogeneous beliefs do not lead to price drift. (*JEL* G12)

1. Introduction

Post-earnings announcement drift and momentum are two of the most intriguing and empirically robust findings about stock-price dynamics.[1] These phenomena imply that, conditional on past performance, future stock returns continue to drift in the same direction. Recent empirical evidence suggests there exists a link between price drift and heterogeneity of beliefs. Specifically, greater disagreement among investors is associated with stronger price drift (e.g., Zhang 2006 and Verardo 2006).

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[1] See Ball and Brown (1968); Jegadeesh (1990); Lehmann (1990); and Jegadeesh and Titman (1993) for early evidence.

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A common, and somewhat casual, explanation for this relationship is based on the intuition that in the presence of noise, prices are slow to aggregate information and, as a result, drift slowly toward the fundamental value. A more involved explanation is based on the notion that in a dynamic rational expectations equilibrium (REE) model, investors may need to forecast the forecasts of others as in Keynes' (1936) "beauty contests." Again, due to noise in investors' signals and in prices themselves, prices aggregate information slowly. In a classic paper, Townsend (1983) argues that in a dynamic REE model with production, agents form beliefs about the beliefs of others and this leads to serial correlation in unconditional forecast errors. More recently, Allen, Morris, and Shin (2006, AMS) examine this issue in an REE model of financial markets. They argue that aggregation of heterogeneous beliefs causes a failure of the law of iterated expectations, which leads to a violation of the martingale property and results in a drift in prices. They relate their finding to that of Keynes (1936), who argued that investors may buy a stock not only because they consider it to be attractive but also because they believe other investors do.

The goal of this paper is to take a closer look at how slow aggregation of heterogeneous beliefs can lead to price drift. Our results challenge the views mentioned above. While heterogeneous beliefs in a multiperiod model indeed make higher-order beliefs relevant, they are not sufficient to generate a price drift. In an REE, agents use the price correctly to eliminate any biases at the aggregate level. Consequently, to obtain a price drift as a result of heterogeneous beliefs, one also needs to assume that investors "agree to disagree" or have "differences of opinions" (DOs) about these beliefs. The contribution of our paper is twofold:

1. We examine what conditions give rise to the phenomenon that AMS had in mind. We show that drift can arise in DO models, since agents may not fully condition on prices. Moreover, while the "standard" DO model, where opinions are common knowledge, delivers drift in a static setting, this is not the case in a dynamic setting. In a dynamic DO model, agents choose to extract information from prices about the beliefs of others. As a result, with either common knowledge or disagreement only about first-order beliefs, there is no price drift. In fact, we show that higher-order differences of opinions are necessary for prices to exhibit drift.
2. In contrast to AMS, we argue that in an REE there is no price drift.[2] As mentioned above, this is because agents condition on the price and correct for any biases. The difference in conclusions stems from the way price drift is defined in each paper. While AMS consider an ex post notion that depends on realized price paths, we argue that one should use an ex ante measure where one conditions only on past and current prices. We further discuss this important distinction in Section 4.

[2] As we discuss in Section 4.2, positive serial correlation in returns may arise in REE models (e.g., Makarov and Rytchkov 2007) when there is serial correlation in the aggregate noise. However, in these cases, drift is driven by the aggregate noise process and does not depend on whether investors have homogeneous beliefs or not.

Our result is consistent with the empirically documented fact that stronger price drift coincides with higher measures of heterogeneity in agents' beliefs. Our paper also contributes to the ongoing debate between REE and DO models. The fact that investors hold heterogeneous beliefs has long been recognized as a key factor in financial markets. The two major paradigms for modeling belief heterogeneity are REE and DO. Both approaches share the view that investors have different valuations, and prices aggregate the different views during the trading process. They differ in whether agents can agree to disagree. In a DO model, agents disagree even when their views become common knowledge. In an REE model, this type of disagreement is ruled out, and investors have different views only if they have private information.

Standard DO models typically assume that the different views are common knowledge (e.g., Harrison and Kreps 1978; Harris and Raviv 1993; Kandel and Pearson 1995). Although primarily made for tractability, this strong assumption is somewhat unnatural. Given the uncertainty regarding fundamentals, it is not clear how investors are certain about other agents' opinions. We show that this common knowledge assumption also eliminates the link between heterogeneous beliefs and price drift. We then relax the common knowledge assumption and instead assume that there is uncertainty about the average opinion. While investors ignore the opinions of others in estimating the value of the asset, they understand that these views may influence intermediate prices, and as such, have a direct impact on their ability to profit from speculating on future prices (see also Cao and Ou-Yang 2005). Investors learn from prices as in an REE, but only to update their beliefs about the average opinion. They trade based on their views about the fundamentals and their beliefs about what other investors think about these fundamentals. We show that in such a model, a drift in prices may arise when investors agree to disagree about the average valuation. Hence, higher-order disagreement is necessary for heterogeneous beliefs to generate price drift.

The rest of the paper is organized as follows. In Section 2, we introduce the basic notation and as a first step compare the classic REE model to a DO model in a static setting. In particular, a DO model naturally leads to price drift as agents do not condition on prices to update their beliefs, but the REE model does not. While the static setup demonstrates why a difference of opinions is necessary for generating price drift, it is not suitable to examine implications of higher-order beliefs. For higher-order beliefs to come into play, one needs to assume that investors live for more than one period.

Section 3 presents the main analysis of the paper. We study a dynamic DO model with long-lived investors who have differences of opinion about the value of the risky asset. In addition, they are also uncertain about the views of other investors. This gives rise to higher-order beliefs, and also creates an REE-type, "learning from prices" feature. While investors may hold strong views about the fundamental value of the asset, they realize that others influence intermediate prices. Hence, each investor infers what others think from the current price to speculate on intermediate prices. Our main result in this section (which is also the main conclusion of the paper) is that in a dynamic setup, higher-order differences of opinions are necessary for heterogeneous beliefs to lead to drift in asset prices. However, if investors do not have differences of opinion about average beliefs, then there is no drift.

Section 4 discusses related literature, starting with a discussion of why our results differ from those in AMS. The two papers have different notions of price drift. We use an ex ante definition that conditions only on information available to agents within the model at the time they make their investment decisions, and we require that higher price changes today be followed by higher price changes on average in the future. AMS, in contrast, use an ex post definition and implicitly conditions on the asset's terminal value, requiring that prices drift toward that realized value. Section 5 concludes. All proofs are in Appendix A.

2. A Static Setup

We begin our analysis by considering a two-date static model based on Grossman (1976) and Hellwig (1980). We argue that heterogeneity of beliefs does not induce price drift in an REE; such drift is present when there are differences of opinion. The model in this section also serves as a benchmark for the dynamic models we examine in the following section. To facilitate a convenient comparison, we begin by describing the features that will be common throughout.

Agents. There is a continuum of investors with utility over terminal wealth at date T and a risk-aversion coefficient γ.

Securities and Trading. There exist a risk-free asset and N risky assets. The net return on the risk-free asset is normalized to zero. The time T liquidation values of the risky assets are given by a vector

$$V \sim N(0, \Sigma_0).$$

Investors can trade at dates $t = 1 \cdots T - 1$. The aggregate supply of the risky assets at date $t < T$ is given by a vector $Z_t = \Sigma_{\tau=1}^{t} z_\tau$, where

$$z_\tau \sim N(0, \Sigma_z).$$

Signals. At the initial trading date $t = 1$, just before trading occurs, each investor i receives a signal S_i about the value of the risky assets V of the form

$$S_i = V + \varepsilon_i,$$

where

$$\varepsilon_i \sim N(0, \Sigma_\varepsilon).$$

We assume that the covariance matrices Σ_0, Σ_ε, and Σ_Z are symmetric and commute with each other. This allows us to generalize our results to a multiasset model while still maintaining analytical tractability.[3]

Information Set. $\mathcal{F}_{i,t}$ denotes agent i's information set at time t.

Notation. $\bar{S}$ denotes $\int_i S_i$. $E_{i,t}[\cdot]$ denotes $E_i[\cdot \mid \mathcal{F}_{i,t}]$, and $\bar{E}_t[\cdot]$ denotes $\int_i E_{i,t}[\cdot]$. $E[\cdot]$ denotes the expectation under the objective distribution.

Since all agents are ex ante identical and have a prior of zero for the liquidation value of the risky assets, $P_0 = 0$ is the price that would prevail if agents were allowed to trade at time $t = 0$. In the static model $T = 2$, the price vector of the risky securities at $T = 2$ is set equal to their liquidation values $P_2 = V$.

The definition for price drift is given below.

Definition 1. Prices exhibit price drift (reversals) if

(i) in a single-asset economy, $E\left[P_2 - P_1 | P_1 - P_0\right]$ is increasing (decreasing, respectively) in $P_1 - P_0$, or

(ii) in a multiple-asset economy, if $P_1^i - P_0^i > P_1^j - P_0^j$ then
$$E\left[P_2^i - P_1^i | P_1 - P_0\right] > E\left[P_2^j - P_1^j | P_1 - P_0\right]$$
$$\left(E\left[P_2^i - P_1^i | P_1 - P_0\right] < E\left[P_2^j - P_1^j | P_1 - P_0\right]\right).$$

Under the symmetry assumptions, if $E\left[P_2 - P_1 | P_1 - P_0\right] = K\left(P_1 - P_0\right)$ for some positive definite (negative definite) K, then prices exhibit drift (reversals).

Agent i, who faces a current price vector P_1, chooses his or her portfolio allocation $X_{i,1}$ by solving

$$X_{i,1} = \arg\max_x E_{i,1}[-\exp(\gamma x'(V - P_1))], \tag{1}$$

which under the joint assumptions of normally distributed payoffs and CARA utility is given by

$$X_{i,1} = \frac{1}{\gamma}\Sigma_V^{-1}(E_{i,1}[V] - P_1), \tag{2}$$

where investor i's posterior beliefs about V are given by

$$V \mid \mathcal{F}_{i,1} \sim N(E_{i,1}[V], \Sigma_V).^4 \tag{3}$$

[3] Commutativity of the covariance matrices implies a factor structure on the random variables, but still allows for correlation across assets (see, for example, Watanabe 2008 and Van Nieuwerburgh and Veldkamp 2006). Symmetry allows us to derive cross-sectional implications from the time-series properties of asset prices.

[4] For ease of exposition, we suppress $\mathcal{F}_{i,1}$ in the notation for variance–covariance matrices throughout. Given that all random variables are assumed to be normally distributed, the variance–covariance matrices are deterministic. The variance–covariance matrices are identical across all agents within a model, but can differ between the REE and DO models.

The difference between the REE and DO models is in the prior beliefs investors have over their information sets $\mathcal{F}_{i,1}$. In an REE, an agent conditions both on the private signal S_i and the price vector P_1. In the DO model, however, agents agree to disagree and each agent conditions only on his or her private signal S_i. This assumption is common to models based on difference of opinions (see, for example, Harrison and Kreps 1978) and is the key departure from a classic REE setup. Each agent believes that no other agent holds information of any additional value to his or her private information. The DO model can alternatively be interpreted as a model with heterogeneous priors where the signals represent these different priors.[5] In contrast, in an REE framework (see, for example, Hellwig 1980), agents believe that $V = \int_i S_i$, and so place a large weight on other agents' information when updating their beliefs about asset values.

Without aggregate noise in the economy, assuming net supply zero implies that the market clearing condition is given by

$$\int_i X_{i,1}(P_1) = 0, \tag{4}$$

so that

$$P_1 = \int_i E_{i,1}[V] \equiv \bar{E}_1[V]. \tag{5}$$

Recall the classic REE result by Grossman (1976), who showed that because the current price is part of the public information that agents use to update their beliefs $P_1 = V$, there is no drift. This is a simple demonstration that in an REE, the intuition described in the introduction is not valid.

In contrast, when agents exhibit difference of opinions, they put more weight on their private signal and less on information held by others, which is reflected in the price. Specifically, in a static model agents choose to ignore the information in prices and set

$$E_{i,1}[V] = \Sigma_V \Sigma_\varepsilon^{-1} S_i, \tag{6}$$

where $\Sigma_V = (\Sigma_0^{-1} + \Sigma_\varepsilon^{-1})^{-1}$. This implies that equilibrium prices are given by

$$P_1 = \Sigma_V \Sigma_\varepsilon^{-1} V, \tag{7}$$

and there is price drift, since

$$E[P_2 - P_1 | P_1 - P_0] = \Sigma_0^{-1} \Sigma_\varepsilon (P_1 - P_0). \tag{8}$$

[5] One can also examine a model that is not as extreme in which agents put a positive weight on others' signals, but assign a higher weight to their own signal. This does not change the qualitative conclusions we draw.

A standard assumption is the existence of noise traders. The addition of noise makes the REE version of the model equivalent to the noisy REE model in Hellwig (1980). In both the REE and DO versions, with aggregate noise, the market clearing condition becomes

$$\int_i X_{i,1}(P_1) = Z_1,$$

where Z_1 is the noisy aggregate supply at time $t = 1$. In this case, the price vector is given by

$$P_1 = \bar{E}_1[V] - \gamma \Sigma_V Z_1. \tag{9}$$

One may conjecture that the introduction of noise would change our conclusions in the REE setup. This is based on the notion that as a result of noise, information is incorporated slowly into prices, leading to a price drift. However, we argue that this again is not true. Noise induces negative correlation in prices. As a result, its presence induces reversals in prices for the REE setup. The static DO model will still exhibit price drift, when the aggregate risky supply Z_1 is not too noisy. Specifically, we argue the following.

Proposition 1. (i) *In REE, prices exhibit reversals;* (ii) *In DO, prices exhibit drift if and only if $\Sigma_Z^{-1}\Sigma_\varepsilon^{-1} - \gamma^2 I > I$, where Σ_Z is the variance of the aggregate supply noise.*

3. A Dynamic Setup

In the previous section, we have shown that while models with REE do not lead to price drift, models with DO may. However, the analysis was restricted to a static setup. In this section, we examine the effect of difference of opinions in a dynamic setup. A key difference is that in a multiperiod model, agents care about the opinions of other agents even when they exhibit DO. This is because the views of other agents affect intermediate prices, which agents can speculate on. As a result, the equilibrium is affected by "beauty contest" considerations, and higher-order beliefs play a key role. When agents do not observe the opinions of others, agents use the information in intermediate prices to learn about the opinions of others, even if they disagree with others. Our main result is that a simple DO structure does not yield a drift in a multiperiod model. A necessary condition in a dynamic model is the presence of higher-order differences of opinions. That is, agents must agree to disagree not only about asset values but also about what the average opinion is.

3.1 Equilibrium prices and price drift

We begin by describing a generic dynamic framework that we shall use to study the effects of different types of disagreements on price drift. In the next subsection, we shall formally define first-order and higher-order disagreements, and argue that while first-order disagreement does not lead to price drift in earlier periods, higher-order disagreement does. In particular, we first extend the static model by one period and set $T = 3$. At date 2, investor i's optimal demand is given by

$$X_{i,2} = \frac{1}{\gamma}\Sigma_V^{-1}(E_{i,2}[V] - P_2), \tag{10}$$

where investor i's beliefs about asset values are given by

$$V \sim N(E_{i,2}[V], \Sigma_V). \tag{11}$$

Market clearing implies that date 2 prices are given by

$$P_2 = \bar{E}_2[V] - \gamma\Sigma_V Z_2. \tag{12}$$

Rolling back, we can show that at date 1, the optimal demand of investor i depends on his or her beliefs about the final payoff V and next period's price P_2. As a result, the price at date 1 is a weighted average of the average beliefs about V and P_2.

Lemma 1. *At date* 1, *the optimal demand of investor i is given by*

$$X_{i,1} = \frac{1}{\gamma}\Sigma_V^{-1}(E_{i,1}[V] - P_1) + \frac{1}{\gamma}\Sigma_{P_2}^{-1}(E_{i,1}[P_2] - P_1), \tag{13}$$

and the price is given by

$$P_1 = \left(\Sigma_V^{-1} + \Sigma_{P_2}^{-1}\right)^{-1}\{\Sigma_V^{-1}\bar{E}_1[V] + \Sigma_{P_2}^{-1}\bar{E}_1[P_2] - \gamma Z_1\}, \tag{14}$$

where investor i's date 1 *beliefs about the price at date* 2 *are given by* $P_2 \mid \mathcal{F}_{i,1} \sim N(E_{i,1}[P_2], \Sigma_{P_2})$.

Because of hedging demands, the optimal demand in Equation (13) consists of two components:[6]

1. *Long-term position.* This is the position that the agent undertakes as a result of his or her long-term view of the asset's final values, $\frac{1}{\gamma}\Sigma_V^{-1}(E_{i,1}[V] - P_1)$.
2. *Speculative position.* This is the position that the agent undertakes as a result of speculating on the next period's prices, $\frac{1}{\gamma}\Sigma_{P_2}^{-1}(E_{i,1}[P_2] - P_1)$.

The price vector P_1 is a weighted average of what agents believe the value of the assets is and what they expect next period's prices to be. Using Equations

[6] A long-term trader generally has a hedging component (unless he or she has logarithmic utility). In the context of an REE, see, for example, Brunnermeier (2001), p. 110.

(12) and (14), we obtain

$$E[P_2 - P_1|P_1 - P_0] = \left(\Sigma_V^{-1} + \Sigma_{P_2}^{-1}\right)^{-1}\Sigma_{P_2}^{-1}E[P_2 - \bar{E}_1[P_2]|P_1 - P_0], \quad (15)$$

implying that price drift exists if and only if

$$E[P_2 - \bar{E}_1[P_2]|P_1 - P_0] \quad (16)$$

is increasing in $P_1 - P_0$.[7]

In the multiple-asset case, the notion of monotonicity is as defined in Definition 1.

3.2 Differing degrees of disagreements

We shall use Equation (16) to study under what conditions heterogeneous beliefs can lead to price drift. In doing so, we will consider different levels of disagreements.

Definition 2. Agents agree on the distribution of a set of random variables if they share a common prior about these variables.

Definition 3. Agents exhibit first-order disagreement about the asset value V if they agree on the distribution of the signals $\{S_i\}$, but not about the joint distribution of $(\{S_i\}, V)$.

Definition 4. Agents exhibit higher-order disagreement if they disagree about both the distribution of signals $\{S_i\}$ and the joint distribution of $(\{S_i\}, V)$.

Hence, a key distinction we will focus on is whether agents disagree only about the relation between their signals and the fundamental value of the asset (i.e., first-order disagreement), or whether they also disagree about the joint distribution of signals (i.e., higher-order disagreement). Note that first-order disagreement leads to disagreement about the fundamental value of the asset, while second-order disagreement leads to disagreement about the beliefs of others and the average valuation.

[7] This is because

$$\begin{aligned}
&E[P_2 - P_1|P_1 - P_0]\\
&= \left(\Sigma_V^{-1} + \Sigma_{P_2}^{-1}\right)^{-1} E[\Sigma_V^{-1}P_2 + \Sigma_{P_2}^{-1}P_2 - \Sigma_V^{-1}\bar{E}_1[V] - \Sigma_{P_2}^{-1}\bar{E}_1[P_2] + \gamma Z_1|P_1 - P_0]\\
&= \left(\Sigma_V^{-1} + \Sigma_{P_2}^{-1}\right)^{-1} E[\Sigma_V^{-1}\bar{E}_2[V] - \gamma Z_2 + \Sigma_{P_2}^{-1}P_2 - \Sigma_V^{-1}\bar{E}_1[V] - \Sigma_{P_2}^{-1}\bar{E}_1[P_2] + \gamma Z_1|P_1 - P_0]\\
&= \left(\Sigma_V^{-1} + \Sigma_{P_2}^{-1}\right)^{-1} \Sigma_{P_2}^{-1}E[P_2 - \bar{E}_1[P_2]|P_1 - P_0].
\end{aligned}$$

3.2.1 First-order disagreements. As Equation (16) reveals, when there are correct expectations about next period's price P_2, we have that $E[P_2 - \bar{E}_1[P_2]|P_1 - P_0] = 0$, and so there is no drift. The price at date 2, P_2, depends on the average opinion across investors $\bar{E}_2[V]$, which depends on the joint distribution of signals $\{S_i\}$, but not on the distribution of V itself. As a result, if investors have correct beliefs about the joint distribution $f(\{S_i\})$, as they do with first-order disagreement, they have correct beliefs about P_2 and there is no drift in prices.

Proposition 2. *If investors exhibit first-order disagreement and are correct about the joint distribution of signals $\{S_i\}$, there is no drift, i.e., $E[P_2 - P_1|P_1 - P_0] = 0$.*

While we focus on disagreements, a similar argument can be used to conclude that there is no drift in an REE model. The standard DO models in the literature are examples of first-order disagreement, since investor signals are assumed to be commonly known. A simple instance of this is as follows.

Example 1. (*Standard DO setup*) *Suppose that V, $U \sim N(0, \Sigma_0)$, and $\varepsilon_{ij} \sim N(0, \Sigma_\varepsilon)$ are all independent. Let investor i's beliefs about his or her signal S_i and about investor j's signal S_j be given by*

$$S_i^i = V + \varepsilon_i \qquad \text{and} \qquad S_j^i = U + \varepsilon_{ij},$$

respectively, where the superscript i is to emphasize that these are i's beliefs. Moreover, the signals are public. This implies that each agent believes his or her signal is informative, but others receive uninformative signals.

Observe that agents do not have the right beliefs about the joint distribution of $\{\{S_i\}, V\}$, but since the signals $\{S_i\}$ are public, investors have the right joint distribution of signals $\{S_i\}$. Furthermore, since by assumption investors have correct beliefs about the noisy risky supply Z_2, $E\left[Z_2 - \bar{E}_1[Z_2]|P_1 - P_0\right] = 0$, and so there is no price drift.

More generally, we have the following corollary.

Corollary 1. *If signals are common knowledge, then there is no price drift.*

The following example shows that even when signals are not commonly known, disagreement can be limited to first-order disagreements.

Example 2. *Suppose that $U \sim N(0, 2)$ and V, $\varepsilon_i \sim N(0, 1)$ are all independent, and let investor i's beliefs about his or her signal S_i and investor j's* *signal S_j be given by*

$$S_i^i = V + \varepsilon_i \qquad \textit{and} \qquad S_j^i = \frac{1}{2}(V + \varepsilon_i) + \frac{1}{2}U + \varepsilon_j,$$

where the superscript i is to emphasize that these are i's beliefs. Moreover, signals are private. In this case, each agent believes that his or her signal is informative and that others receive noisy versions of his or her own signal.

A direct computation shows that $cov(S_i^i, S_j^i) = cov(S_k^i, S_j^i) = 1$ and $var(S_i^i) = var(S_j^i) = 2$, which implies that investors agree about the joint distribution of signals $\{S_i\}$. Agent i also believes that S_j is not informative about V, given his or her information. Again, since investors agree on the joint distribution of $\{S_i\}$, prices do not exhibit drift.

3.2.2 Higher-order disagreement and price drift. While rational expectations or standard difference of opinions models do not generate price drift, heterogeneous information settings can still lead to price drift. Below we characterize the conditions under which a model with higher-order disagreement can lead to drift in prices.

Recall that we have shown that prices exhibit drift when $E[P_2 - \bar{E}_1[P_2]|P_1 - P_0]$ is increasing in $P_1 - P_0$. In a DO model, Equation (12) can be written as

$$P_2 = \bar{E}_2[V] - \gamma\Sigma_V Z_2 = \Sigma_V \Sigma_\varepsilon^{-1} \bar{S} - \gamma\Sigma_V Z_2, \tag{17}$$

so that

$$\begin{aligned} E[P_2 - \bar{E}_1[P_2]|P_1 - P_0] = \Sigma_V \Sigma_\varepsilon^{-1} E[\,\bar{S} - \bar{E}_1[\,\bar{S}\,]|P_1 - P_0] \\ -\gamma\Sigma_V E[Z_2 - \bar{E}_1\,[Z_2]\,|P_1 - P_0]. \end{aligned} \tag{18}$$

While investors may or may not have noisy private information about $\bar{S}$, they are assumed not to have any private information about the noisy supply shocks Z_2. As a result, the only source of information about supply shocks are prices (i.e., $E_{i,1}\,[Z_2] = E_{i,1}\,[Z_1|P_1 - P_0]$).

If investors believe that prices are less informative than they actually are, their conditional expectations put too little weight on prices, and this implies that both $E[Z_2 - \bar{E}_1\,[Z_2]\,|P_1 - P_0]$ and $E[\,\bar{S} - \bar{E}_1\left[\,\bar{S}\,\right]|P_1 - P_0]$ increase in $P_1 - P_0$. But since $\Sigma_V \Sigma_\varepsilon^{-1}$ is positive definite and $-\gamma\Sigma_V$ is negative definite, the two terms in (18) have opposite effects. However, when the aggregate supply noise is small, one can characterize a sufficient condition for prices to exhibit drift.

Lemma 2. *As $\Sigma_Z \to 0$, prices exhibit price drift if and only if $E[\,\bar{S}\,|P_1 - P_0] - \bar{E}_1[\,\bar{S}\,|P_1 - P_0] = \kappa(P_1 - P_0)$ for some $\kappa > 0$.*

When the supply shock noise is small, whether or not there is price drift depends only on the difference in the expectations of $\bar{S}$ conditional on $P_1 - P_0$ under the objective distribution and the investors' distribution. If investors put too little weight on the price when updating their beliefs about the aggregate signal, then prices exhibit price drift. We see this effect in the following two simple examples.

Example 3. *Consider a setup similar to Example* 2, *except that investors exhibit higher-order disagreement. Specifically, suppose each investor believes that he or she knows the average signal, U, but other investors do not.*[8] *A direct computation shows that investor i's beliefs about prices are*

$$P_2 = \frac{1}{4}(S_i + U) - \frac{\gamma}{2}Z_2 \quad \textit{and} \quad P_1 = \frac{a}{2}(S_i + U) + bZ_1,$$

where a is positive and b is negative, so that his or her conditional expectation at time 1 of P_2 is

$$E_{i,1}[P_2] = \frac{1}{4}(S_i + U) - \frac{\gamma}{2}\frac{1}{b}\left(P_1 - \frac{a}{2}(S_i + U)\right) = \frac{1}{4}(S_i + U) - \frac{\gamma}{2}Z_1.$$

However, the objective distributions are given by

$$P_2 = \frac{1}{2}V - \frac{\gamma}{2}Z_2 \quad \textit{and} \quad P_1 = aV + bZ_1.$$

This implies that Equation (16) *is given by*

$$\begin{aligned} &E[P_2 - \bar{E}_1[P_2]|P_1 - P_0] \\ &\quad = E\left[\frac{1}{2}V - \frac{\gamma}{2}Z_1 - \left(\frac{1}{4}(S_i + U) - \frac{\gamma}{2}Z_1\right)|P_1 - P_0\right] = \frac{1}{4}E[V|P_1 - P_0], \end{aligned}$$

and prices exhibit drift.

The advantage in Example 3 is that it is simple to follow; however, the structure may look special. The following example considers a more general structure. Specifically, it generalizes the standard DO model in Example 1 to a setup in which investors agree to disagree about the average valuation as a result of private information about the aggregate signal of other investors.

Example 4. *Consider a setup similar to Example* 1, *except that investors also exhibit higher-order disagreement. Suppose that $V, U, W \sim N(0, \Sigma_0)$, ε_i, ε_j,*

[8] This example can be extended to the case where investors do not believe they know U perfectly. In particular, to the case where each investor has a private signal T_i about U, where investor i's beliefs about his or her signal T_i and about investor j's signal T_j are given by $T_i^i = U + \eta_i$ and $T_j = \frac{1}{2}(T_i + \omega) + \eta_j$, where $\omega \sim N(0, \Sigma_\omega)$. Specifically, investor i believes that others' signals are not informative.

$\eta_i, \eta_j \sim N(0, \Sigma_\varepsilon)$ are all independent, and investor i's beliefs about his or her signal S_i and investor j's signal S_j are given by

$$S_i^i = V + \varepsilon_i \qquad \text{and} \qquad S_j^i = U + \varepsilon_j.$$

Moreover, these signals are private. In addition, suppose investors also receive private signals T_i about the aggregate signal U. In particular, investor i's beliefs about his or her signal T_i and investor j's signal T_j are given by

$$T_i^i = U + \eta_i, \qquad T_j^i = W + \eta_j,$$

and these signals are private too. Supply shocks are distributed as $Z_t = \sum_{\tau=1}^{t} z_\tau$, where $z_\tau \sim N(0, \Sigma_z)$. As in Example 1, each agent believes that his or her signals are informative and that others receive uninformative signals. This is true for both signals about the fundamental value V and signals about the aggregate signal U.

Again, agents disagree not only about the distribution of $(\{S_i, T_i\}, V)$, but also about the distribution of $\{S_i, T_i\}$, and hence exhibit higher-order disagreement. In this case, when supply shocks are not too large, prices exhibit price drift.

Claim 1. *In Example 4, as $\Sigma_Z \to 0$ prices exhibit price drift if $\Sigma_0(\Sigma_\varepsilon + \Sigma_V) > \mathrm{I}$.*

3.3 Multiperiod extensions and degrees of disagreement

Within a three-date ($T = 3$) dynamic economy, Examples 3 and 4 show that higher-order disagreement can lead to price drift, while Proposition 2 shows that with first-order disagreement there is no price drift. Based on these results, a natural conjecture would be that as we add additional periods we would require higher levels of disagreement to attain price drift. Somewhat surprisingly, this is not true.[9]

Consider adding one more period, so that $T = 4$. Agents' optimal demands and the price functions in the last two trading periods (i.e., $t = 2$ and $t = 3$) correspond to the ones identified in Section 3.1. Specifically, in the last trading period, investor i' s optimal demand is given by

$$X_{i,3} = \frac{1}{\gamma}\Sigma_V^{-1}(E_{i,3}[V] - P_3), \tag{19}$$

and prices are given by

$$P_3 = \bar{E}_3[V] - \gamma\Sigma_V Z_3. \tag{20}$$

[9] We thank an anonymous referee for suggesting we look into this question. Our discussion of second-order or higher-order disagreement is casual and somewhat informal. We do not define explicitly what an nth order disagreement is. For a more complete analysis, one needs to follow a formal construction of belief hierarchies such as the one in Mertens and Zamir (1985) or Brandenburger and Dekel (1993).

At date 2, demand is given by

$$X_{i,2} = \frac{1}{\gamma}\Sigma_V^{-1}(E_{i,2}[V] - P_2) + \frac{1}{\gamma}\Sigma_{P_3}^{-1}(E_{i,2}[P_3] - P_2), \tag{21}$$

and prices are given by

$$P_2 = \left(\Sigma_V^{-1} + \Sigma_{P_3}^{-1}\right)^{-1}\left\{\Sigma_V^{-1}\bar{E}_2[V] + \Sigma_{P_3}^{-1}\bar{E}_2[P_3] - \gamma Z_2\right\}. \tag{22}$$

As we show in Appendix A, this implies the following result.

Lemma 3. *At date* 1*, the optimal demand for investor i is given by*

$$X_{i,1} = \frac{1}{\gamma}\Sigma_{P_2}^{-1}(E_{i,1}[P_2] - P_1) + \frac{1}{\gamma}\left(\Sigma_{P_3}^{-1} + \Sigma_V^{-1}\right)(E_{i,1}[V] - P_1), \tag{23}$$

and the price is given by

$$P_1 = \left[\Sigma_{P_2}^{-1} + \left(\Sigma_{P_3}^{-1} + \Sigma_V^{-1}\right)\right]^{-1}\left\{\left(\Sigma_{P_3}^{-1} + \Sigma_V^{-1}\right)\bar{E}_1[V] + \Sigma_{P_2}^{-1}\bar{E}_1[P_2] - \gamma Z_1\right\}, \tag{24}$$

where investor i's date 1 *beliefs about the price at date* 2 *are given by $P_2 \mid \mathcal{F}_{i,1} \sim N(E_{i,1}[P_2], \Sigma_{P_2})$.*

When agents receive signals about V only at time 0 and agree to disagree about V, we have that $E_{i,3}[V] = E_{i,2}[V] = E_{i,1}[V] = \Sigma_V \Sigma_\varepsilon^{-1}\bar{S}$. This implies that when noisy supply shocks are small, second-order disagreement may be enough to generate price drift.

Lemma 4. *Suppose investors receive signals about V at time 0 and agree to disagree about it. As $\Sigma_Z \to 0$,*

$$\begin{aligned} E[P_2 - P_1 | P_1 - P_0] = {} & A E[P_2 - \bar{E}_1[P_2] | P_1 - P_0] \\ & + B E[\bar{E}_2[\bar{S}] - \bar{S} | P_1 - P_0], \end{aligned} \tag{25}$$

for A and B positive. This implies that second-order disagreement may be sufficient to generate price drift.

The logic applied in Section 3.2.1 seems to suggest that with only second-order disagreement, these two expressions should equal zero. However, the arguments used in Section 3.2.1 do not directly apply. To see why, note that our definitions of higher-order disagreement are based on agents' ex ante beliefs, which they have before observing prices. However, since agents condition on prices in equilibrium, higher-order disagreement may arise endogenously. For the individual investor, prices are signals about other investors' beliefs. Observing these prices may lead investors to disagree on the distribution of

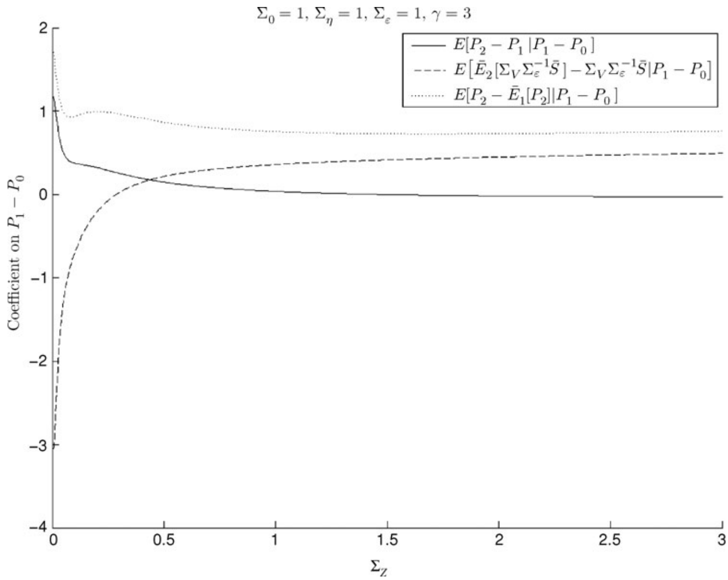

Figure 1
Four-date ($T = 4$) model with second-order disagreement
This figure exhibits momentum for small values of Σ_z (less than 1.455), but reversals for large values of Σ_z. The parameters of the model are set to $\Sigma_0 = 1,\ \Sigma_\eta = 1,\ \Sigma_\varepsilon = 1$, and $\gamma = 3$. The solid line plots the coefficient of $E\,[P_2 - P_1|P_1 - P_0]$ on $P_1 - P_0$, the dashed line plots the coefficient of $E[\bar{E}_2[\Sigma_V\,\Sigma_\varepsilon^{-1}\bar{S}] - \Sigma_V\,\Sigma_\varepsilon^{-1}\bar{S}|P_1 - P_0]$, and the dotted line plots the coefficient of $E[P_2 - \bar{E}_1\,[P_2]|P_1 - P_0]$.

variables that they agreed on ex ante. Consider a simple example where investors agree on the distribution of X ex ante, but agree to disagree about the distribution of Y. If investors form their beliefs using a signal that depends on both (e.g., $Z = X + Y$), then they may also disagree about the conditional distribution of X.

Note that prices in earlier periods depend on the average opinion about V. When investors exhibit second-order disagreement, they agree to disagree about this average opinion. As a result, when investors condition on the price to update their expectations about higher-order beliefs, they will disagree about these beliefs in equilibrium. When supply shocks are not very noisy, this implies that second-order disagreement may be sufficient to generate price drift in the four-date model. Appendix A formalizes this intuition in more detail, and Figure 1 shows an example of a four-date model with second-order disagreement in which prices exhibit price drift for small levels of supply noise (when $\Sigma_Z < 1.45$).

The above intuition may appear to contradict the result in Proposition 2, where we argued that if investors exhibit first-order disagreements then there will be no price drift. But this is not true. The logic above does not apply in this earlier case, as the only random variable about which agents agree to disagree is the value of the asset itself. However, note that the prices are not a function of the asset's value, but are functions of the average beliefs about the value, and so with first-order disagreements, we cannot have price drift.

4. Discussion and Related Literature

Our objective has been to evaluate whether slow aggregation of information/ opinions can lead to price drift. To get a comprehensive understanding of the matter at hand, we have analyzed both REE and DO settings. The philosophical debate about which of these paradigms is more appropriate has been fierce over the years, but has largely remained inconclusive.[10] A part of this debate has centered on whether DO models can be supported in a stationary, repeated setting, where investors can learn how to use prices over time. This issue is beyond the scope of our model, which is stylized and has a finite horizon. However, Acemoglu, Chernozhukov, and Yildiz (2006) have shown that in an environment in which individuals are uncertain about how they should interpret signals, individuals with different priors may never agree, even after observing the same infinite sequence of signals. Moreover, while DO models are often identified as "behavioral" or "irrational," this need not be the case. For instance, Aumann (1976) argues in his classic work that rational people are likely to agree to disagree. Moreover, the recent interest in behavioral economics has led to a renewed interest in DO models (e.g., Scheinkman and Xiong 2003), especially, since allowing for such disagreements not only has intuitive appeal but also seems useful in generating trading volume.

We do not have a stake in the RE versus DO debate at large, and we have simply analyzed the feasibility of the proposed mechanism under both approaches. For supporters of the REE models, our paper shows that the slow aggregation of information is not an appropriate channel to generate price drift. For supporters of DO models, our analysis demonstrates that, conceptually, DO models may generate price drift through a slow aggregation of opinions channel. However, we have also highlighted the fact that relaxing the common knowledge assumption is a necessary condition. Without differences in higher-order beliefs, DO models fail to generate price drift as well.

4.1 Allen, Morris, and Shin (2006)

Our paper is most closely related to that of Allen, Morris, and Shin (2006), who argue that price drift may arise in an REE model as a result of higher-order beliefs. The starting point for both papers is the fact that in a dynamic model, higher-order beliefs become relevant. However, we arrive at very different conclusions regarding the implications for the resulting price patterns. AMS argue that higher-order beliefs may lead to price drift in REE models. We argue that since in an REE the public information available to agents includes the price, such patterns do not arise. Moreover, with noise in the economy, prices

[10] A discussion of the relevance of the noncommon prior assumption appears in Morris (1995), for example.

instead exhibit reversals. We have shown that only models in which agents agree to disagree can make the AMS intuition valid as agents put less weight on prices. It is important to note that we do not argue that the mathematical results in AMS are incorrect; instead, we disagree with their interpretation of these results.

To see why we arrive at a different conclusion from AMS, begin by considering the example that appears in the first part of their paper. This example presents a statistical exercise that provides the intuition for the REE model. Agents are interested in estimating a random variable V. All agents have common priors about the distribution of V, given by $V \sim N(0, 1/\rho_V)$. Agents observe two signals: (i) a public signal, Y, and (ii) a private signal, S_i. The signals are normally distributed, with $Y = V + \delta$ and $S_i = V + \varepsilon_i$, where $\{\varepsilon_i\}$ are i.i.d. and independent of δ; the distributions are given by $\varepsilon_i \sim N(0, 1/\rho_\varepsilon)$ and $\delta \sim N(0, 1/\rho_\delta)$. In this case, we have

$$E_i[V] \equiv E[V|S_i, Y] = \frac{\rho_\varepsilon}{\rho_\varepsilon + \rho_\delta + \rho_V} S_i + \frac{\rho_\delta}{\rho_\varepsilon + \rho_\delta + \rho_V} Y + \frac{\rho_V}{\rho_\varepsilon + \rho_\delta + \rho_V} 0. \tag{26}$$

Hence,

$$\bar{E}[V] \equiv \int_i E_i[V] = \frac{\rho_\varepsilon}{\rho_\varepsilon + \rho_\delta + \rho_V} \bar{S} + \frac{\rho_\delta}{\rho_\varepsilon + \rho_\delta + \rho_V} Y = \frac{\rho_\varepsilon}{\rho_\varepsilon + \rho_\delta + \rho_V} V + \frac{\rho_\delta}{\rho_\varepsilon + \rho_\delta + \rho_V} Y, \tag{27}$$

where the second equality follows from the fact that with a continuum of agents we have that $\bar{S} = V$.[11]

The average estimate is indeed biased, as it puts excessive weight on the public signal Y. As AMS note, this feature breaks the martingale property and may generate positive autocorrelation or price drifts. If one defines $H_T = V$ and $H_t = \bar{E}[H_{t+1}]$, then one can show that $H_t = \alpha_t Y + \beta_t V$, where α_t is decreasing in t and β_t is increasing in t. Given this representation, it is tempting to conclude that similar reasoning may lead to drift in an REE. We argue this is not the case because, in an REE, the public signal is the price of the asset. Agents observe the price that reflects their average opinion and use it to correct any bias. In terms of the example above, this implies that Y is a function of H_t, and hence one cannot simply look at the dynamics of α_t and β_t. The simple model we have examined in Section 2 demonstrates this effect in a static setup.

In the second part of AMS, they consider a standard REE model in which they argue that a price drift arises; they base this claim on Propositions 1

[11] The exact structure in AMS is a slightly reduced version of this example. Specifically, $V \sim N(y, 1/\rho_V)$, where y, the ex ante mean of V, is exogenously specified.

and 2 of their paper. While they do not formally define price drift, following Proposition 2 they suggest that a situation in which the price approaches the true fundamental value in incremental steps would have many outward appearances of momentum in prices.

To see why conditioning on the ex post realized final value is problematic, consider a price process that is a martingale; the price in each period equals the expected value of the asset conditional on current information. If the information sets are increasing over time, then conditional on the final value we find that the price slowly converges to this value. Still, such a price process does not exhibit a price drift, as martingale differences are uncorrelated and unpredictable. Moreover, many standard information-based models exhibit such behavior. In particular, the price process in the multiperiod model in Kyle (1985) and Glosten and Milgrom (1985) satisfies Propositions 1 and 2 of AMS.

Appendix B shows that if we apply what we believe is the proper definition of a price drift to the AMS model, then there is no price drift, and instead prices exhibit reversals. These price reversals are due purely to the mean reversion in the noise terms, as prices would exhibit reversals even with homogeneous information.

4.2 Other literature

Our paper is also closely related to the "beauty contest" metaphor described by Keynes (1936). Keynes based his analogy on contests that were popular in England at the time, where a newspaper would print 100 photographs, and people would write in and say which six faces they liked most. Everyone who picked the most popular face was automatically entered in a raffle, where they could win a prize. Given these incentives, people would not necessarily choose faces they found the prettiest, but instead would choose those they believed would catch the fancy of the other competitors, all of whom were using the same logic in making their choices. This led agents to form higher-order beliefs. The link to financial markets follows from the fact that prices reflect some average opinion of different investors. Since it is possible to resell the stock, it may not be enough for investors to pick the stock they find most attractive, as they must also consider which stocks others will find attractive. As a result, investors need to form beliefs about the average valuation, the average opinion about the average valuation, and so on; in doing so they engage in higher-order reasoning. Indeed, AMS show, formally, in a finite horizon economy that the price t periods before the last one reflects the t-th order average opinion. Keynes' intuition seems more appropriate to a DO model, since in such a model agents have different fundamental valuations. In contrast, investors in an REE model agree that there is an objective fundamental value and cannot agree to disagree, even if they have different information.

Other papers, such as that of Makarov and Rytchkov (2007), consider models of asymmetric information in which nontrivial higher-order beliefs potentially lead to possible correlations in price changes. However, in Makarov and

Rytchkov (2007), positive serial correlation arises only under a very specific noise structure that is correlated itself. Hence, in their model, heterogeneity in investor beliefs is not the source of drift in prices.[12]

While we focus on heterogeneity of beliefs as a potential explanation, a number of alternative explanations for price drift have been proposed. Broadly, they fall into two categories: behavioral and rational. In behavioral models, some or all of the agents in the economy exhibit specific cognitive biases that lead to underreaction. For example, in Barberis, Shleifer, and Vishny (1998), agents exhibit conservatism and representativeness biases, while in Daniel, Hirshleifer, and Subrahmanyam (1998), agents overestimate the precision of their signals and suffer from a self-attribution bias. Hong and Stein (1999) assume the presence of "news watchers" who receive public signals slowly but do not use the price to update their beliefs. These behavioral biases lead to an underreaction to public information, and so lead to price drift.

The rational explanations are either risk-based or information-based. In risk-based models, the drift is a result of the dynamics of the underlying fundamentals. In Berk, Green, and Naik (1999), the potential driving force is variation of exposure over the life cycle of a firm's endogenously chosen investment projects.[13] In Johnson (2002), momentum potentially arises as an artifact of stochastic expected growth rates of a firm's cash flows.[14] In Holden and Subrahmanyam (2002), momentum is a result of increased precision of information over time. Sequential arrival of information prior to the terminal date leads to a decrease in the risk borne by the market because the mass of informed traders increases over time. As a consequence, there is a gradual decrease in the conditional risk premium required to absorb liquidity shocks.

As shown, to potentially generate price drift in a DO setting through a slow aggregation of opinions channel, one needs to depart from the common knowledge assumption that is typically made in DO models. Biais and Bossaerts (1998) show how one can relax the common knowledge assumption and still avoid the infinite regress problem. Varian (1989) considers a fully revealing equilibrium in which agents have different priors and receive subsequent information. Ottaviani and Sorensen (2006) analyze pari-mutuel betting markets with heterogeneous priors and private information. Allen, Morris, and Postlewaite (1993) have both differences in priors and differences in information interacting. In their model, a necessary condition for a strong bubble

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[12] Kondor (2004) shows, in a rational expectations framework, how differences in higher-order beliefs can lead to high trading volume and volatile prices, following public announcements. Cao and Ou-Yang (2005) demonstrate, in a difference of opinions setting, how failure of the law of iterated expectations for average beliefs can cause prices to be higher (lower) than the price that any individual trader would be willing to pay for the asset if he or she was precluded to trade again in subsequent periods.

[13] Gomes, Kogan, and Zhang (2003) relax the partial equilibrium restrictions in Berk, Green, and Naik (1999) analyzing a related problem in general equilibrium.

[14] Sagi and Seasholes (2006) depart from Johnson (2002) by including observable firm attributes in the determinants of cashflows. They show that a firm's revenues, costs, and growth options combine to determine the dynamics of its return autocorrelation.

to occur is that agents' trades are not common knowledge. Kraus and Smith (1998) consider a model in which multiple self-fulfilling equilibria arise because of uncertainty about other investors' beliefs. They term this "endogenous sunspots." It is shown that such sunspots can lead to asset prices being higher than in the equilibrium with common knowledge. Other papers that explore the role of higher-order beliefs include Rubinstein (1989); Abel and Mailath (1994); and Shin (1996).

5. Conclusion

We have analyzed the potential link between heterogeneous beliefs and price drift, and demonstrated that to generate price drift as an outcome of slow aggregation of heterogeneous beliefs, it is necessary to have higher-order disagreement.

Our analysis highlights the fact that in order for slow aggregation of information to be a viable channel for generating price drift, one needs to consider models with difference of opinions. In a rational expectations equilibrium, there is no price drift. The intuition presented in AMS and other papers that relates higher-order beliefs to price drift holds only when agents' public information is completely exogenous and, in particular, does not include prices. However, in REE models with heterogeneous beliefs, prices are part of agents' information sets. All agents update their beliefs about asset values using asset prices, and these prices are endogenously determined to reflect the average opinion of the agents. As a result, a price drift does not arise in REE models, contrary to what has been suggested in the literature.

Standard DO models assume common knowledge of agents' opinions—each agent knows what others believe about the fundamental value of the asset, and agents "agree to disagree." We relax this common knowledge assumption and assume that agents are uncertain about the beliefs of others. Interestingly, we show that in a dynamic framework, price drift is not robust in a setting with only first-order disagreement since prices satisfy a martingale property in all but the last period. This is because in earlier rounds of trade, the equilibrium is similar to a classic REE, except investors infer the opinions of others from the price instead of from their private information. To obtain a price drift, one also needs differences in opinions about higher-order beliefs. Specifically, it is not sufficient to break the common priors assumption, but it is also necessary to relax the assumption that agents have common knowledge of the heterogeneous priors.

In the past few years, DO models have received increased interest in financial economics. We have focused on price drift, but more generally our analysis highlights the importance of exploring the consequences of learning about other agents' priors. Models that relax the common knowledge assumption also introduce interesting empirical challenges, as testing such models requires an identification of higher-order disagreement.

Appendix A

Notation. In general, denote investor i's date t posterior beliefs about variable Y as $Y|\mathcal{F}_{i,t} \sim N(E_{i,t}[Y], \Sigma_Y)$, and let $E[Y]$ and $E_i[Y]$ denote the expected value of Y under the objective distribution and investor i's beliefs, respectively.

Proof of Proposition 1. (i) For the REE case,

$$\begin{aligned} E\,[V - P_1|P_1] &= E[\,V - \left(\bar{E}\,[V] + \gamma\Sigma_V Z_1\right)\big|\, P_1] \\ &= \int_i E\,[\,V - E_i\,[V|S_i, P_1]|\, P_1] + \gamma\Sigma_V E\,[\,Z_1|\,P_1] \\ &= \gamma\Sigma_V E\,[\,Z_1|\,P_1] = \gamma\Sigma_V \kappa P_1, \end{aligned} \tag{A1}$$

for some $\kappa < 0$, where the third equality follows from the fact that since agents correctly incorporate the current price in forming beliefs about the future price, and prices are linear, the law of iterated expectation holds for every agent, and averaging over all i does not change this.

Under the assumptions of symmetry that we make, $E\,[P_2 - P_1|P_1 - P_0] = M\,(P_1 - P_0)$ for some positive definite (negative definite) M implies that prices exhibit drift (reversals). Since the covariance matrices are symmetric, this implies that prices satisfy the multiasset definition of reversals.

(ii) For the DO case, let the noise $Z_1 \sim N\,(0, \Sigma_Z)$, then the price takes the form

$$P_1 = \bar{E}[V] - \gamma\Sigma_V Z_1 = \Sigma_V \Sigma_\varepsilon^{-1} V - \gamma\Sigma_V Z_1, \tag{A2}$$

which implies that

$$E\,[V|P_1] = [\gamma^2 \Sigma_V \Sigma_0^{-1} \Sigma_V + \Sigma_V \Sigma_\varepsilon^{-1} \Sigma_Z^{-1} (\Sigma_V \Sigma_\varepsilon^{-1})']^{-1} \Sigma_V \Sigma_\varepsilon^{-1} \Sigma_Z^{-1} P_{1.} \tag{A3}$$

Thus,

$$\begin{aligned} E\,[V - P_1|P_1] &= \left[\gamma^2 \Sigma_V \Sigma_0^{-1} \Sigma_V + \Sigma_V \Sigma_\varepsilon^{-1} \Sigma_Z^{-1} \left(\Sigma_V \Sigma_\varepsilon^{-1}\right)'\right]^{-1} \Sigma_V \Sigma_\varepsilon^{-1} \Sigma_Z^{-1} P_1 - P_1 \\ &= \left[\gamma^2 \Sigma_0^{-1} + \Sigma_\varepsilon^{-1} \Sigma_Z^{-1} \Sigma_\varepsilon^{-1\prime}\right]^{-1} \left(\Sigma_0^{-1} + \Sigma_\varepsilon^{-1}\right) \Sigma_\varepsilon^{-1} \Sigma_Z^{-1} P_1 - P_1 \\ &= \left[\gamma^2 \Sigma_0^{-1} + \Sigma_\varepsilon^{-1} \Sigma_Z^{-1} \Sigma_\varepsilon^{-1\prime}\right]^{-1} \Sigma_0^{-1} \left(\Sigma_\varepsilon^{-1} \Sigma_Z^{-1} - \gamma^2\right) P_{1.} \end{aligned}$$

Hence, when $\Sigma_\varepsilon^{-1} \Sigma_Z^{-1} - \gamma^2 I \ > 0$, $E\,[V - P_1|P_1] = \kappa P_1$ for some κ positive definite, which, under the symmetry assumption, implies that prices exhibit drift. ■

Proof of Lemma 1. At date 1, investor i's optimal demand is given by

$$\begin{aligned} X_{i,1} &= \arg\max_x E_{i,1}[\exp\{-\gamma x'(P_2 - P_1) - \gamma X_{i,2}'(V - P_2)\}] \\ &= \arg\max_x E_{i,1}\big[\exp\{-\gamma x'(P_2 - P_1) - (1/2)(E_{i,1}[V] - P_2)'\Sigma_V^{-1}(E_{i,1}[V] - P_2)\}\big] \\ &= \arg\max_x E_{i,1}\big[\exp\{-\gamma x'(-\chi + E_{i,1}[V] - P_1) - (1/2)\chi'\Sigma_V^{-1}\chi\}\big], \end{aligned} \tag{A4}$$

where $\chi \equiv E_{i,2}[V] - P_2$ and agent i's date 1 beliefs about P_2 are $P_2 \mid \mathcal{F}_{i,1} \ \sim N(E_{i,1}[P_2], \Sigma_{P_2})$. We know that if $X \sim N(\mu, \Sigma)$, and $Q^* = X'AX + a'X + d$, and if A is symmetric,

$$M_{Q^*}(1) = |I - 2A\Sigma|^{-1/2} \exp\left[-\frac{1}{2}(\mu'\Sigma^{-1}\mu - 2d) + \frac{1}{2}(\mu + \Sigma a)'(I - 2A\Sigma)^{-1}\Sigma^{-1}(\mu + \Sigma a)\right]. \tag{A5}$$

We apply the above result by substituting $A = -(1/2)\Sigma_V^{-1}$, $a = \gamma x$, $d = \gamma x'(P_1 - E_{i,1}[V])$, $\mu = E_{i,1}[V] - E_{i,1}[P_2]$, and $\Sigma = \Sigma_{P_2}$, and by noting that only d and a depend on the variable of interest, we have that the FOC of Equation (A4) is given by

$$\frac{\partial d}{\partial x} + (\mu + \Sigma a)'(I - 2A\Sigma)^{-1}\Sigma^{-1}\Sigma\frac{\partial a}{\partial x} = 0,$$

$$(P_1 - E_{i,1}[V])' + (E_{i,1}[V] - E_{i,1}[P_2] + \gamma\Sigma_{P_2}x)'\left(I + \Sigma_V^{-1}\Sigma_{P_2}\right)^{-1} = 0, \tag{A6}$$

which implies that the optimal demand is given by

$$X_{i,1} = \frac{1}{\gamma}\Sigma_V^{-1}(E_{i,1}[V] - P_1) + \frac{1}{\gamma}\Sigma_{P_2}^{-1}(E_{i,1}[P_2] - P_1), \tag{A7}$$

which implies the price is given by

$$P_1 = \left(\Sigma_V^{-1} + \Sigma_{P_2}^{-1}\right)^{-1}\left\{\Sigma_V^{-1}\bar{E}_1[V] + \Sigma_{P_2}^{-1}\bar{E}_1[P_2] - \gamma Z_1\right\}. \tag{A8}$$

■

Proof of Proposition 2. The case for the REE model follows immediately from the fact that when all the investors are correct in their beliefs, the law of iterated expectations holds for each investor i:

$$E[P_2 - E_{i,1}[P_2]|P_1] = 0.$$

In the difference of opinions model, note that investor i's beliefs about date 2 price depend on his or her beliefs about the distribution of the signals $\{S_i\}$, and not on his or her beliefs about V. In particular, P_2 is measurable with respect to $\{\{S_i\}, Z_2\}$. When investors have first-order difference of opinions, their beliefs about the joint distribution $f(\{S_i\})$ are correct, and so their beliefs about the joint distribution $f(\{S_i\}, Z_2)$ are correct. This, in turn, implies that for each agent i, the law of iterated expectations holds: $E[P_2 - E_{i,1}[P_2]|P_1] = 0$. ■

Proof of Lemma 2. In the limit when $\Sigma_Z = 0$, $E[Z_2 - \bar{E}_1[Z_2]|P_1] = 0$ since Z_2 is known. Since we are only considering linear equilibria, without loss of generality, we can denote the true distribution of the date 1 price as $P_1 = A\bar{S} + BZ_1$, and investor i's beliefs about the date 1 price as $P_1 = A_i\bar{S} + B_iZ_1 + W$, where $W \sim N(0, \Sigma_W)$ is independent of the other random variables. Furthermore, denote investor i's beliefs about $\bar{S}$ before observing $P_1$by $\bar{S} \sim N(U_i, \Sigma_U)$, where $\int_i U_i = \lambda\bar{S}$ for some $0 \leq \lambda \leq I$ and $\Sigma_U < \Sigma_0$. This reflects the fact that investors may have private information about $\bar{S}$. Then the expected value of Z_2 conditional on P_1 under the objective distribution and investor i's beliefs is given by

$$E[Z_2|P_1] = LB^{-1}P_1 \text{ and } E_{i,1}[Z_2] = L_iB_i^{-1}(P_1 - A_iU_i), \tag{A9}$$

respectively, where $L = [B\Sigma_ZB' + A\Sigma_0A']^{-1}B\Sigma_ZB'$, $L_i = [B_i\Sigma_ZB_i' + A_i\Sigma_0A_i' + \Sigma_W]^{-1}$ $B_i\Sigma_ZB_i'$. Similarly, expected value of $\bar{S}$ conditional on P_1 under the objective distribution and investor i's beliefs is given by

$$E\left[\bar{S}\,|P_1\right] = MA^{-1}P_1 \text{ and } E_i[\bar{S}\,|\mathcal{F}_{i,1}] = M_iA_i^{-1}P_1 + (1 - M_i)U_{i,} \tag{A10}$$

where $M = I - L$ and $M_i = [B_i\Sigma_ZB_i' + A_i\Sigma_UA_i' + \Sigma_W]^{-1}A_i\Sigma_UA_i'$. Note that investor i's beliefs about $\bar{S}$ can be rewritten as

$$E_i[\bar{S}\,] = \omega E_i[\bar{S}\,|P_1] + (1 - \omega)\lambda^{-1}U_{i,} \tag{A11}$$

for some $0 \leq \omega \leq I$, where $E_i[\bar{S}\,|P_1] = [B_i\Sigma_ZB_i' + A_i\Sigma_0A_i' + \Sigma_W]^{-1}A_i\Sigma_0A_i'(A_i^{-1}P_1)$. This implies that the first term in the expression for price drift depends only on the difference in the expectations of $\bar{S}$ conditional on P_1 under the objective distribution and the investor's distribution:

$$\begin{aligned} E[\bar{S} - \bar{E}_1\left[\bar{S}\right]|P_1] &= E[\bar{S} - \omega \bar{E}_1[\bar{S}|P_1] + (I-\omega)\bar{S}|P_1] \\ &= \omega\{E[\bar{S}|P_1] - \bar{E}_1[\bar{S}|P_1]\}. \end{aligned} \tag{A12}$$

Next, note that if the price coefficients are bounded away from zero, as $\Sigma_Z \to 0$, we have that $L \to 0$, $L_i \to 0$, $M \to I$, and $M_i \to [A_i \Sigma_U A_i' + \Sigma_W]^{-1} A_i \Sigma_U A_i'$, which implies that

$$E[Z_2 - \bar{E}_1\left[Z_2\right]|P_1] \to 0. \tag{A13}$$

But A and A_i must be bounded away from zero as long as investors have some private information about $\bar{S}$. Also, B and B_i must also be bounded away from zero as $\Sigma_Z \to 0$ since investors have less uncertainty about the supply shock, and so prices respond to the shocks more accurately. This implies that when $\Sigma_Z \to 0$, prices exhibit price drift if and only if $E[\bar{S}|P_1] - \bar{E}_1[\bar{S}|P_1] = \kappa P_1$ for some κ positive definite. ■

Proof of Claim 1. Investors believe that the date 1 price is of the form

$$P_1 = AU + BZ_1 + CW,$$

which implies that

$$\begin{aligned} E_{i,1}\left[U\right] &= M_i A^{-1} P_1 + (I - M_i)\left(\Sigma_0^{-1} + \Sigma_\eta^{-1}\right)^{-1} \Sigma_\eta^{-1} T_i \\ E_{i,1}\left[Z_2\right] &= L_i B^{-1}\left(P_1 - A\left(\Sigma_0^{-1} + \Sigma_\eta^{-1}\right)^{-1} \Sigma_\eta^{-1} T_i\right), \end{aligned} \tag{A14}$$

where

$$\begin{aligned} M_i &= \left[A\left(\Sigma_0^{-1} + \Sigma_\eta^{-1}\right)^{-1} A' + B\Sigma_z B' + C\Sigma_0 C'\right]^{-1} A\left(\Sigma_0^{-1} + \Sigma_\eta^{-1}\right)^{-1} A', \\ L_i &= \left[A\left(\Sigma_0^{-1} + \Sigma_\eta^{-1}\right)^{-1} A' + B\Sigma_z B' + C\Sigma_0 C'\right]^{-1} B\Sigma_z B'. \end{aligned}$$

However, the objective distribution is of the form

$$P_1 = (A+B)U + CZ_1,$$

so that

$$\begin{aligned} E\left[U|P_1\right] &= M\left(A+C\right)^{-1} P_{1,} \\ E\left[Z_2|P_1\right] &= LB^{-1} P_{1,} \end{aligned} \tag{A15}$$

where

$$\begin{aligned} M &= [(A+C)\,\Sigma_0\,(A+C)' + B\Sigma_Z B']^{-1}\,(A+C)\,\Sigma_0\,(A+C)\,, \\ L &= I - M. \end{aligned}$$

Recall that the date 2 price is of the form

$$P_2 = \Sigma_V \Sigma_\varepsilon^{-1} U - \gamma \Sigma_V Z_2,$$

where $\Sigma_V = (\Sigma_0^{-1} + \Sigma_\varepsilon^{-1})^{-1}$.

This implies that the date 1 price is implicitly given by the expression

$$\left(\Sigma_V^{-1} + \Sigma_{P_2}^{-1} - \Sigma_{P_2}^{-1}\left(\Sigma_V \Sigma_\varepsilon^{-1} M_i A^{-1} - \gamma \Sigma_V L_i B^{-1}\right)\right) P_1 \tag{A16}$$
$$= \Sigma_\varepsilon^{-1} U - \gamma Z_1 + \Sigma_{P_2}^{-1}\left(\Sigma_V \Sigma_\varepsilon^{-1}(I - M_i) + \gamma \Sigma_V L_i B^{-1} A\right)\left(\Sigma_0^{-1} + \Sigma_\eta^{-1}\right)^{-1} \Sigma_\eta^{-1} W,$$

where $\Sigma_{P_2} = \Sigma_V \Sigma_\varepsilon^{-1}(1 - M_i)(\Sigma_0^{-1} + \Sigma_\eta^{-1})^{-1}(\Sigma_V \Sigma_\varepsilon^{-1})' + \gamma^2 \Sigma_V ((1 - L_i)\Sigma_Z + \Sigma_Z)\Sigma_V$. The condition for price drift is given by

$$E[P_2 - \bar{E}_1[P_2]|P_1] = \Sigma_V \begin{bmatrix} \Sigma_\varepsilon^{-1} M (A + C)^{-1} - \gamma L B^{-1} - \Sigma_\varepsilon^{-1} M_i A^{-1} \\ +\Sigma_\varepsilon^{-1}(I - M_i)\left(\Sigma_0^{-1} + \Sigma_\eta^{-1}\right)^{-1} \Sigma_\eta^{-1} M (A + C)^{-1} \\ +\gamma L_i B^{-1}\left(I - A\left(\Sigma_0^{-1} + \Sigma_\eta^{-1}\right)^{-1} \Sigma_\eta^{-1} M (A + C)^{-1}\right) \end{bmatrix} P_{1.} \tag{A17}$$

When the supply noise $\Sigma_Z \to 0$, note that $M \to I$, $L \to 0$, $L_i \to 0$, and

$$M_i \to \left[A\left(\Sigma_0^{-1} + \Sigma_\eta^{-1}\right)^{-1} A' + C \Sigma_0 C'\right]^{-1} A\left(\Sigma_0^{-1} + \Sigma_\eta^{-1}\right)^{-1} A'.$$

From the expression for P_1 in Equation (A16), we know that

$$\left(\Sigma_V^{-1} + \Sigma_{P_2}^{-1} - \Sigma_{P_2}^{-1} \Sigma_V \Sigma_\varepsilon^{-1} M_i A^{-1}\right) P_1 = \Sigma_\varepsilon^{-1} U - \gamma Z_1 + \Sigma_V^{-1} \Sigma_\varepsilon \Sigma_\eta^{-1} W,$$

which implies $A = \Sigma_V \Sigma_\varepsilon^{-1} \Sigma_\eta \Sigma_\varepsilon^{-1} C = \Sigma_V \Sigma_\varepsilon^{-1} C$ and $M_i = [\Sigma_\varepsilon^{-1} \Sigma_V \Sigma_\varepsilon^{-1} + \Sigma_0]^{-1} \Sigma_\varepsilon^{-1} \Sigma_V \Sigma_\varepsilon^{-1}$ so that Equation (A17) reduces to

$$\begin{aligned} &E[P_2 - \bar{E}_1[P_2]|P_1] \\ &\to \Sigma_V \Sigma_\varepsilon^{-1}\left[\left(\Sigma_V \Sigma_\varepsilon^{-1} + I\right)^{-1} - M_i\left(\Sigma_V \Sigma_\varepsilon^{-1}\right)^{-1} + (I - M_i)\Sigma_V \Sigma_\varepsilon^{-1}\left(\Sigma_V \Sigma_\varepsilon^{-1} + I\right)^{-1}\right] C^{-1} P_1 \\ &= \Sigma_V \Sigma_\varepsilon^{-1}\left(\Sigma_V \Sigma_\varepsilon^{-1} + I\right)^{-1}\left[\Sigma_\varepsilon^{-1} \Sigma_V \Sigma_\varepsilon^{-1} + \Sigma_0\right]^{-1}\left[\Sigma_0 - \Sigma_\varepsilon^{-1} + \Sigma_0 \Sigma_V \Sigma_\varepsilon^{-1}\right] C^{-1} P_1 \\ &= \Sigma_V \Sigma_\varepsilon^{-1}\left(\Sigma_V \Sigma_\varepsilon^{-1} + I\right)^{-1}\left[\Sigma_\varepsilon^{-1} \Sigma_V \Sigma_\varepsilon^{-1} + \Sigma_0\right]^{-1} \Sigma_\varepsilon^{-1}[\Sigma_0 \Sigma_\varepsilon + \Sigma_0 \Sigma_V - I] C^{-1} P_1, \end{aligned}$$

implying a price drift since $\Sigma_0 \Sigma_\varepsilon + \Sigma_0 \Sigma_V > I$. ■

Proof of Lemma 3. Using the results of Equations (19)–(22) implies that the date 1 optimal demand for investor i is given by

$$\begin{aligned} X_{i,1} = \arg\max_x E_{i,1}\Big[\exp\left\{-\gamma x'(P_2 - P_1)\right\} \\ E_{i,2}\left[\exp\left\{\begin{array}{l} -\gamma X_{i,2}'(-\left(E_{i,3}[V] - P_3\right) + E_{i,2}[V] - P_2) \\ -(1/2)\left(E_{i,3}[V] - P_3\right)' \Sigma_V^{-1}\left(E_{i,3}[V] - P_3\right) \end{array}\right\}\right]\Big]. \end{aligned} \tag{A18}$$

Let $\theta_1 = E_{i,2}[V] - P_2$, $\theta_2 = E_{i,2}[P_3] - P_2$, and

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \sim N\left(\begin{pmatrix} E_{i,1}[V] - E_{i,1}[P_2] \\ E_{i,1}\left[E_{i,2}[P_3]\right] - E_{i,1}[P_2] \end{pmatrix}, \begin{pmatrix} \Sigma_{P_2} & \Sigma_{P_2} \\ \Sigma_{P_2} & \Sigma_{P_2} + \Sigma_{P_3,1} \end{pmatrix}\right),$$

where $\mathrm{var}_1\left[E_{i,2}\left[P_3\right]\right]=\Sigma_{P_3,1}$, $a=\begin{pmatrix}\gamma x\\0\end{pmatrix}$, $d=-\gamma x'\left(E_{i,2}[V]-P_1\right)$, $A=\begin{pmatrix}K & -\Sigma_{P_3}^{-1}\\0 & 0\end{pmatrix}$ and

$$K=\left[\frac{1}{2}(I+\Sigma_{P_3}\Sigma_V^{-1})'(I+\Sigma_V^{-1}\Sigma_{P_3})^{-1}\Sigma_{P_3}^{-1}(I+\Sigma_{P_3}\Sigma_V^{-1})-\Sigma_V^{-1}\right]=\frac{1}{2}\left(\Sigma_{P_3}^{-1}-\Sigma_V^{-1}\right).$$

Then, the optimal demand is given by

$$X_{i,1}=\arg\max_x E_{i,1}\left[\exp\left\{d+a'\theta+\theta' A\theta\right\}\right]. \tag{A19}$$

As before, note that only a and d depend on x, and so the first-order condition is given by

$$\frac{\partial d}{\partial x}+(\mu_\theta+\Sigma_\theta a)'(I-2A\Sigma_\theta)^{-1}\Sigma_\theta^{-1}\Sigma_\theta\frac{\partial a}{\partial x}=0. \tag{A20}$$

This implies that

$$P_1-E_{i,1}\left[P_2\right]+\left(P_1-E_{i,1}[V]\right)2\left(\Sigma_{P_3}^{-1}-K\right)\Sigma_{P_2}+\gamma\Sigma_{P_2}x=0,$$

which implies that the optimal demand is given by

$$X_{i,1}=\frac{1}{\gamma}\Sigma_{P_2}^{-1}\left(E_{i,1}\left[P_2\right]-P_1\right)+\frac{1}{\gamma}\left(\Sigma_{P_3}^{-1}+\Sigma_V^{-1}\right)\left(E_{i,1}[V]-P_1\right). \tag{A21}$$

Market clearing then implies

$$P_1=\left[\Sigma_{P_2}^{-1}+\left(\Sigma_{P_3}^{-1}+\Sigma_V^{-1}\right)\right]^{-1}\left\{\left(\Sigma_{P_3}^{-1}+\Sigma_V^{-1}\right)\bar{E}_1[V]+\Sigma_{P_2}^{-1}\bar{E}_1\left[P_2\right]-\gamma Z_1\right\}. \tag{A22}$$

■

Proof of Lemma 4. When agents receive signals about V only at time 0 and agree to disagree about V, we have that $E_{i,3}\left[V\right]=E_{i,2}[V]=E_{i,1}[V]=\Sigma_V\Sigma_\varepsilon^{-1}\bar{S}$. This implies that $P_3=\Sigma_V\Sigma_\varepsilon^{-1}\bar{S}-\gamma\Sigma_V Z_3$, and prices in the first two trading dates are given by

$$P_2=(\Sigma_V^{-1}+\Sigma_{P_3}^{-1})^{-1}\{\Sigma_V^{-1}\left(\Sigma_V\Sigma_\varepsilon^{-1}\bar{S}\right)+\Sigma_{P_3}^{-1}\bar{E}_2[\Sigma_V\Sigma_\varepsilon^{-1}\bar{S}-\gamma\Sigma_V Z_3]-\gamma Z_2\}, \tag{A23}$$

$$P_1=\left[\Sigma_{P_2}^{-1}+\left(\Sigma_{P_3}^{-1}+\Sigma_V^{-1}\right)\right]^{-1}\left\{\left(\Sigma_{P_3}^{-1}+\Sigma_V^{-1}\right)\Sigma_V\Sigma_\varepsilon^{-1}\bar{S}+\Sigma_{P_2}^{-1}\bar{E}_1\left[P_2\right]-\gamma Z_1\right\}. \tag{A24}$$

Equations (A23) and (A24) imply that the price drift condition depends on

$$\begin{aligned}&[\Sigma_{P_2}^{-1}+(\Sigma_{P_3}^{-1}+\Sigma_V^{-1})]E[P_2-P_1|P_1-P_0]\\&\quad=E[\Sigma_{P_2}^{-1}P_2+\Sigma_V^{-1}(\Sigma_V\Sigma_\varepsilon^{-1}\bar{S})+\Sigma_{P_3}^{-1}\bar{E}_2[\Sigma_V\Sigma_\varepsilon^{-1}\bar{S}-\gamma\Sigma_V Z_3]-\gamma Z_2-\Sigma_{P_2}^{-1}\bar{E}_1[P_2]\\&\qquad+(\Sigma_{P_3}^{-1}+\Sigma_V^{-1})\Sigma_V\Sigma_\varepsilon^{-1}\bar{S}-\gamma Z_1|P_1-P_0].\end{aligned} \tag{A25}$$

Assuming $\Sigma_Z\to 0$ implies the condition becomes

$$\begin{aligned}&\Sigma_{P_2}^{-1}E\left[P_2-\bar{E}_1\left[P_2\right]|\;P_1-P_0\right]+\Sigma_{P_3}^{-1}E\left[\bar{E}_2[\Sigma_V\Sigma_\varepsilon^{-1}\bar{S}]-\Sigma_V\Sigma_\varepsilon^{-1}\bar{S}|P_1-P_0\right]\\&\quad=\Sigma_{P_2}^{-1}(\Sigma_V^{-1}+\Sigma_{P_3}^{-1})^{-1}\left\{\Sigma_V^{-1}E\left[\Sigma_V\Sigma_\varepsilon^{-1}\bar{S}-\bar{E}_1\left[\Sigma_V\Sigma_\varepsilon^{-1}\bar{S}\right]|P_1-P_0\right]\right.\\&\qquad\left.+\Sigma_{P_3}^{-1}E\left[\bar{E}_2[\Sigma_V\Sigma_\varepsilon^{-1}\bar{S}]-\bar{E}_1\left[\bar{E}_2[\Sigma_V\Sigma_\varepsilon^{-1}\bar{S}]\right]|P_1-P_0\right]\right\}\\&\qquad+\Sigma_{P_3}^{-1}E\left[\bar{E}_2\left[\Sigma_V\Sigma_\varepsilon^{-1}\bar{S}\right]-\Sigma_V\Sigma_\varepsilon^{-1}\bar{S}|P_1-P_0\right].\end{aligned}$$

Recall that if investors have first-order difference of opinions, then $E_{i,2}\left[\bar{S}\right]$ is correct for all investors, and so $E\left[\Sigma_V \Sigma_\varepsilon^{-1}\bar{S} - \bar{E}_2[\Sigma_V \Sigma_\varepsilon^{-1}\bar{S}]|P_2\right] = 0$, which implies there is no price drift.

Now suppose that investors have second-order disagreement. This implies that beliefs about $\bar{S}$ are not correct, but the unconditional beliefs about $\bar{E}_2[\Sigma_V \Sigma_\varepsilon^{-1}\bar{S}]$ are correct for all investors—let us denote $\bar{E}_2[\Sigma_V \Sigma_\varepsilon^{-1}\bar{S}]$ by W. Note that this does not imply that conditional beliefs about W are correct. For example, suppose investors condition on the price $P_1 = A\bar{S} + BW$. If their beliefs about $\bar{S}$ are incorrect, then their conditional beliefs about W, $E_i[W|P_1]$ are not equal to the objective conditional expectation $E[W|P_1]$. Even if conditional beliefs were correct (which they are not), date 1 prices could still exhibit drift since $E[\Sigma_V \Sigma_\varepsilon^{-1}\bar{S} - \bar{E}_1[\Sigma_V \Sigma_\varepsilon^{-1}\bar{S}]|P_1] \neq 0$ and $E[\bar{E}_2[\Sigma_V \Sigma_\varepsilon^{-1}\bar{S}] - \Sigma_V \Sigma_\varepsilon^{-1}\bar{S}|P_1] \neq 0$. So, even with only second-order disagreement, we may still have price drift in a four-date model. ■

Appendix B

In this appendix, we show that the REE model considered in AMS does not lead to a price drift and instead leads to price reversals. Let us apply our definition of drift to the AMS setup. In their setting, they have overlapping generations of myopic investors. Agent i at time t lives for one period and observes a signal

$$S_{it} = V + \varepsilon_{it},$$

where $\varepsilon_{it} \sim N(0, 1/\beta)$. The time T liquidation value of the asset is given by

$$h = E_0[V],$$

and supply shocks are $z_t \sim N(0, 1/\gamma_t)$.

Setting $P_0 = h$, agent i's information set is given by $\{P_0, P_1, ..., P_t, S_{it}\}$.When supply shocks are i.i.d., the price at time t is given by $P_t = \bar{E}_t\bar{E}_{t+1}\cdots\bar{E}_T(V) - \frac{Var_t(P_{t+1})}{\tau}z_t$. Therefore,

$$\begin{aligned} E\left[P_2 - P_1|P_1 - P_0\right] &= E\left[\bar{E}_2\cdots\bar{E}_T(V) - \frac{Var_2(P_3)}{\tau}z_2 - P_1\middle| P_1 - P_0\right] \\ &= E\left[\bar{E}_2\cdots\bar{E}_T(V) - \bar{E}_1\bar{E}_2\cdots\bar{E}_T(V) + \frac{Var_1(P_2)}{\tau}z_1\middle| P_1 - P_0\right] \\ &= E\left[H_2 - \bar{E}_1 H_2\middle| P_1 - P_0\right] + \frac{Var_1(P_2)}{\tau}E\left[z_1\middle| P_1 - P_0\right], \end{aligned} \quad \text{(B1)}$$

where $H_2 = \bar{E}_2\cdots\bar{E}_T(V)$. The first part is zero by the law of iterated expectations, since

$$\begin{aligned} E\left[H_2 - \bar{E}_1 H_2\middle| P_1 - P_0\right] &= E\left[\int_i \left(H_2 - E_{i,1}H_2\right)\middle| P_1 - P_0\right] \\ &= \int_i E\left[H_2 - E\left[H_2|P_1 - P_0, S_{it}\right]\middle| P_1 - P_0\right] = 0. \end{aligned} \quad \text{(B2)}$$

Since z_1 is negatively correlated with P_1, $\frac{Var_1(P_2)}{\tau}E\left[z_1| P_1 - P_0\right] = k(P_1 - P_0)$ for some $k < 0$, and so the price change $P_2 - P_1$ is negatively correlated to lagged price change $P_1 - P_0$.

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